

3+1 Formulations of Einstein's equations

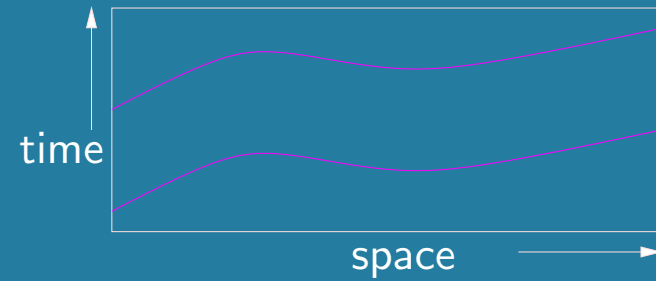
Mark A. Scheel
Caltech

Outline:

- Quick summary of last talk.
- 3+1 formulation (i.e. how one writes equations) is not unique.
 - ★ Changing formulation changes numerical results for fixed numerical solution scheme.
- Hyperbolic formulations.

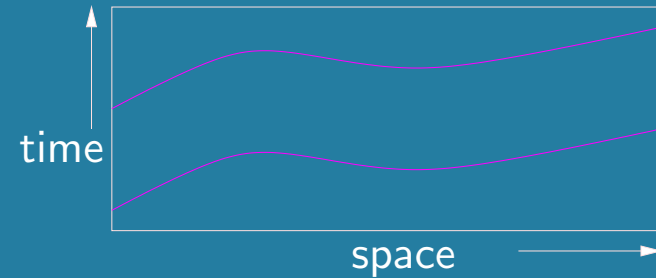
Summary of Part I

- Cauchy evolution of black hole spacetimes:



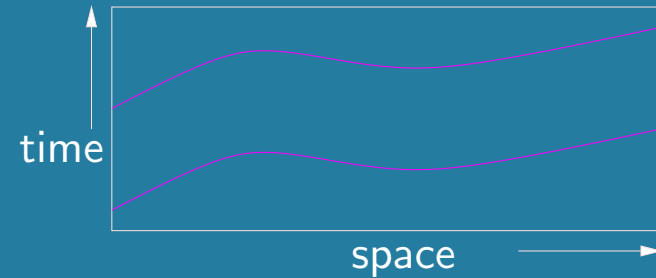
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 - * Evolution equations.



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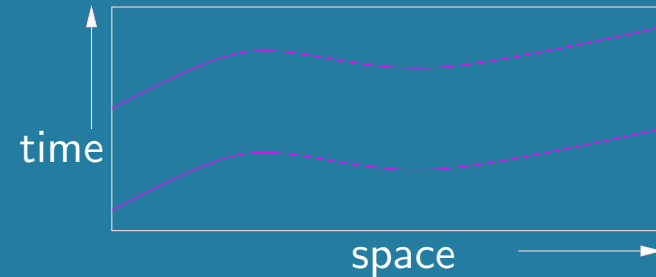
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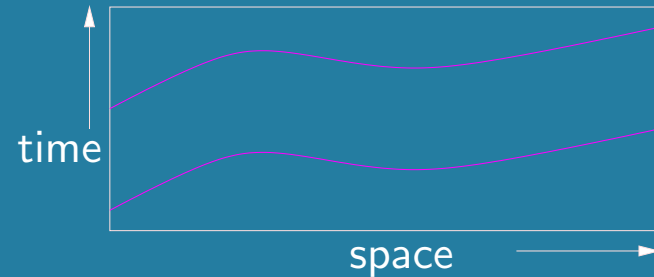
- Why so difficult?

- ★ Must avoid physical singularities (horizon excision).
- ★ Difficult to find good coordinate system.
- ★ “Almost all” coordinate systems become singular.
- ★ Constraints violated by free evolution.

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- ★ Difficult to find good coordinate system.
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- ★ Constraints violated by free evolution.
- ★ Are 3+1 evolution equations well-behaved analytically?

Problem: Unphysical Solutions

- Einstein **evolution** equations have unphysical solutions.
 - ★ Constraint-violating solutions
 - * Hope to control using variational methods.
 - ★ Pure gauge (coordinate) solutions
 - * Present even when constraints satisfied.

Problem: Unphysical Solutions

- Einstein **evolution** equations have unphysical solutions.
 - ★ Constraint-violating solutions
 - * Hope to control using variational methods.
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 - * Present even when constraints satisfied.
- Unphysical solutions can grow quickly.
- Unphysical solutions can propagate acausally.
(EM: Jackson Problem 6.19)

3+1 formulation is not unique

- ADM in 1st order form: Variables $g_{ij}, K_{ij}, d_{kij} \equiv \partial_k g_{ij}$

$$\mathcal{C} = \bar{R}^i{}_i + K^i{}_i K^i{}_i - K_{ij} K^{ij} = 0$$

$$\mathcal{C}_i = D^j K_{ij} - D_i K^j{}_j = 0$$

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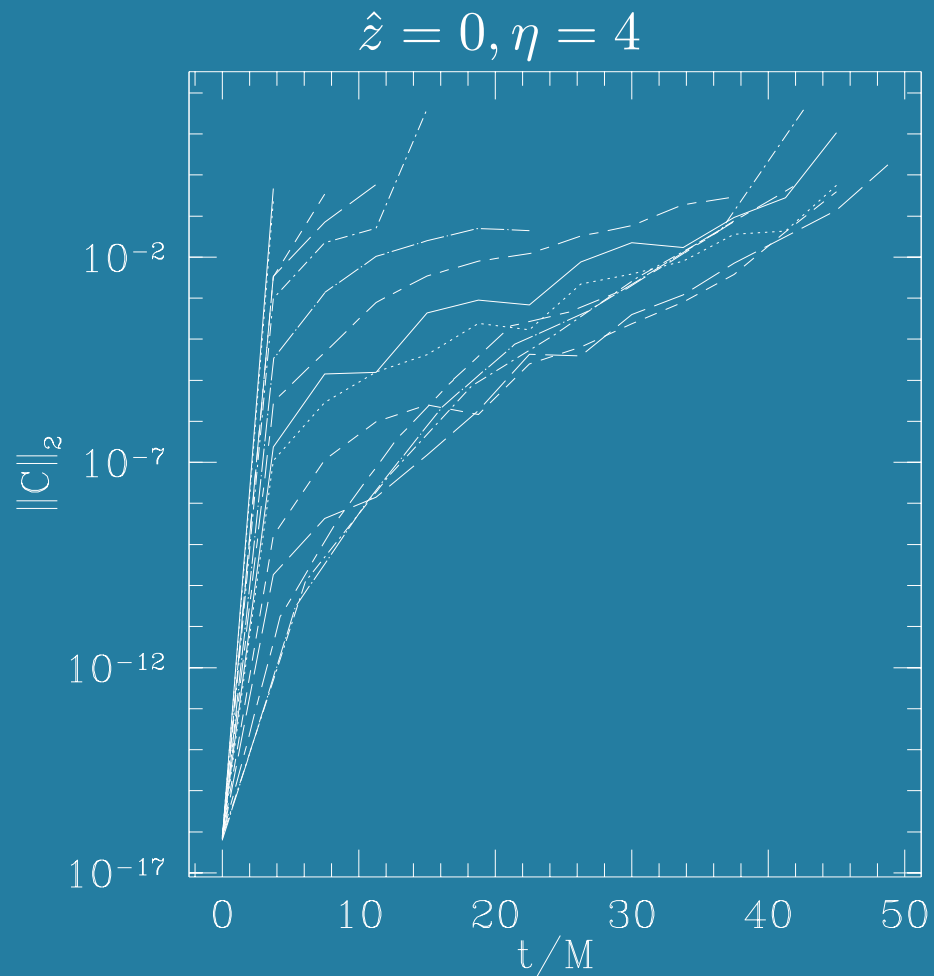
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- Can add constraints.
- Can also change variables, e.g. $P_{ij} = K_{ij} + \hat{z} g_{ij} K_{mn} g^{mn}$
- Does not change physical solutions.
- Does change unphysical solutions of *evolution* system (gauge modes, constraint-violating modes)

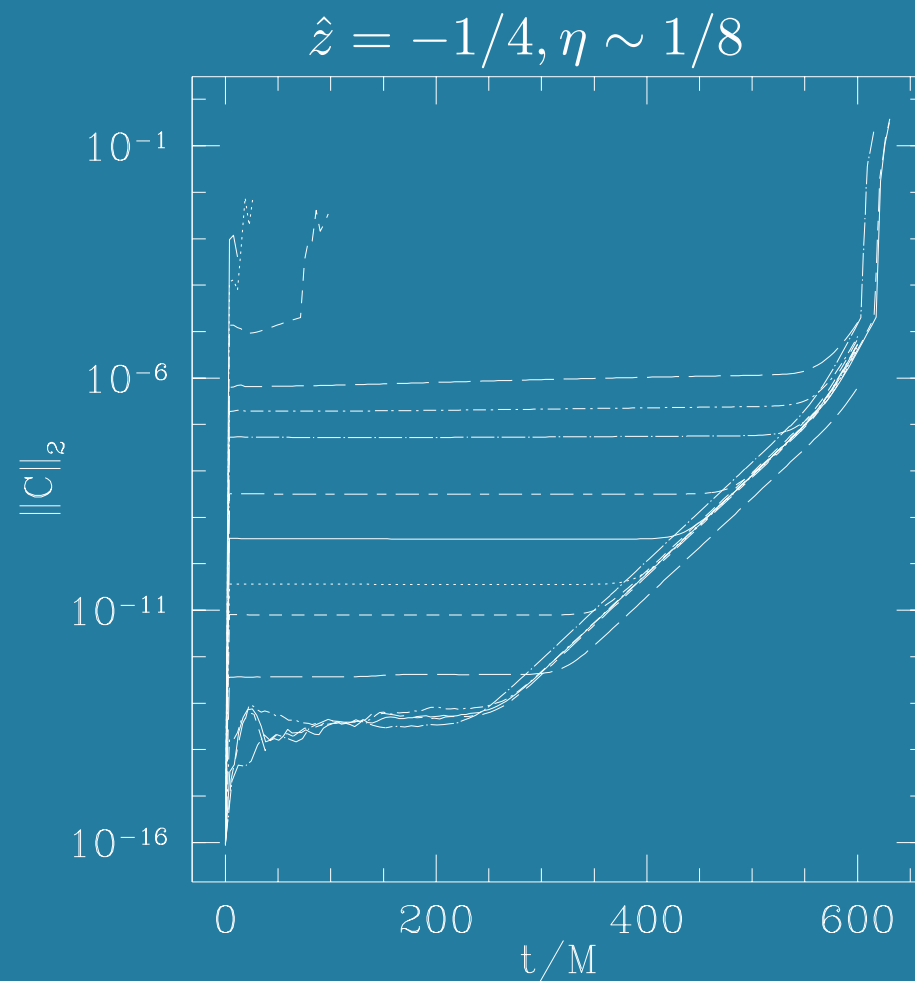
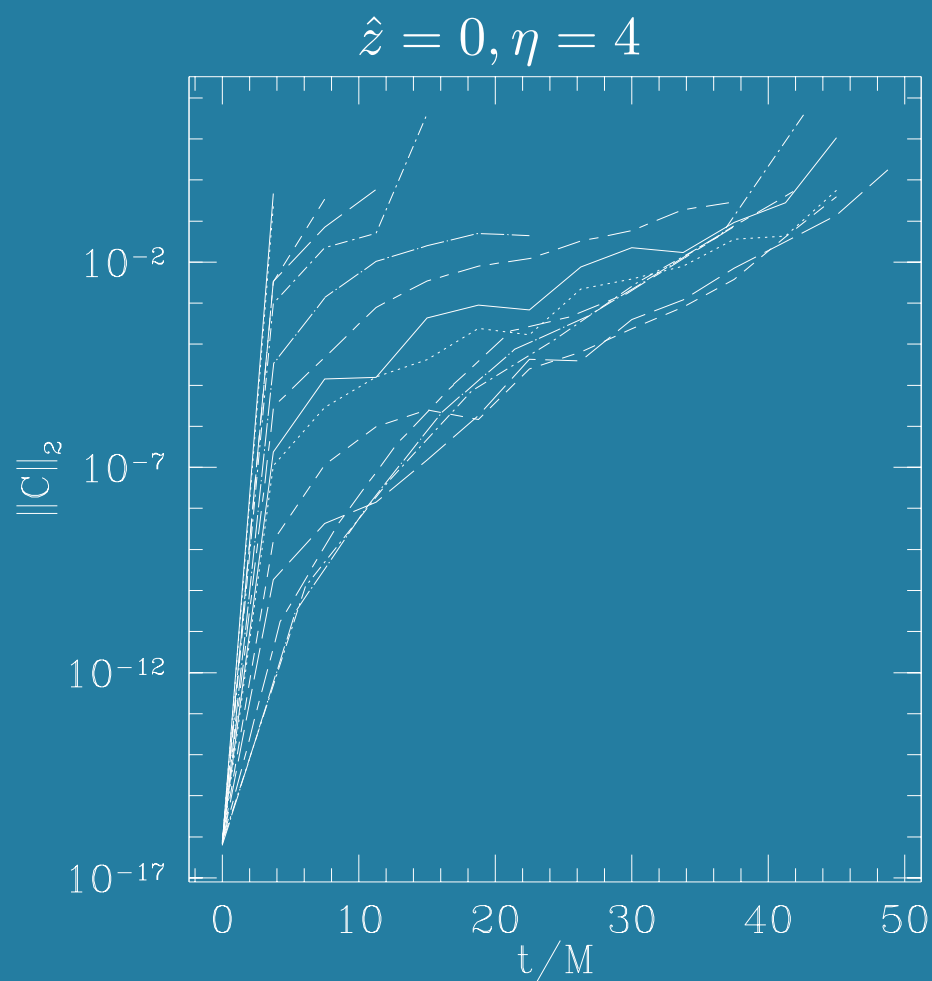
Formulation affects numerical results (3D)

- Fully 3D numerical evolution of Schwarzschild BH.



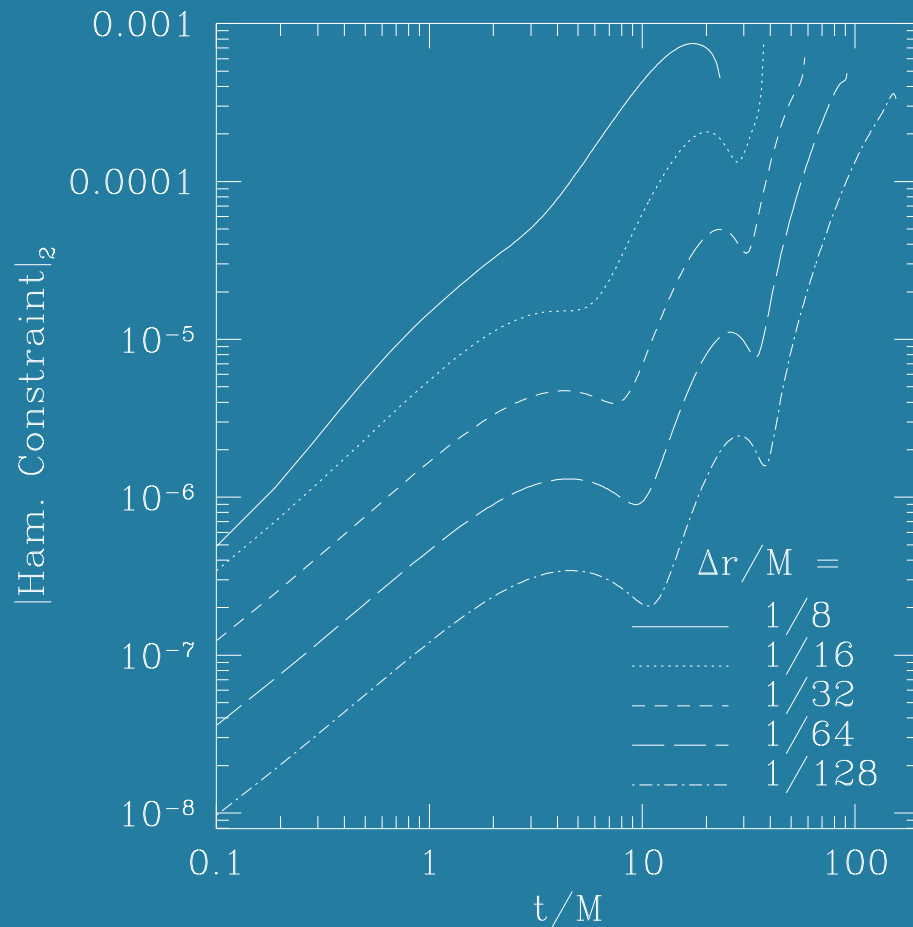
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- Fully 3D numerical evolution of Schwarzschild BH.
- Change **formulation** parameters \hat{z} , η . Numerics **unchanged**.



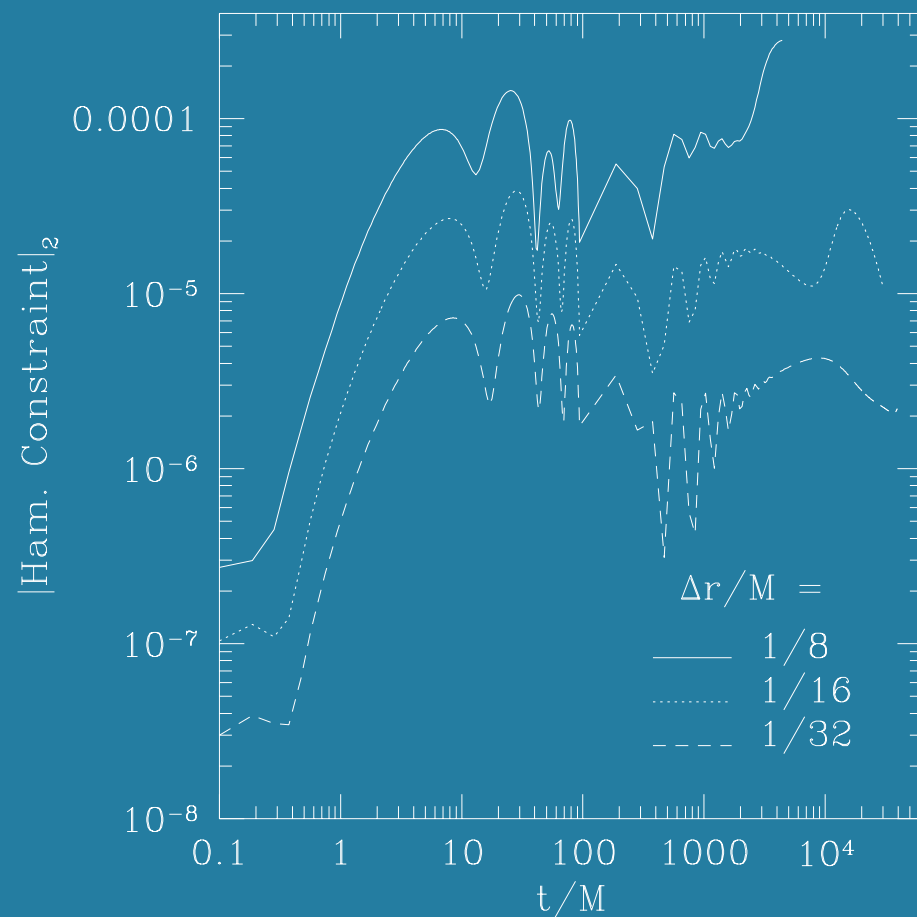
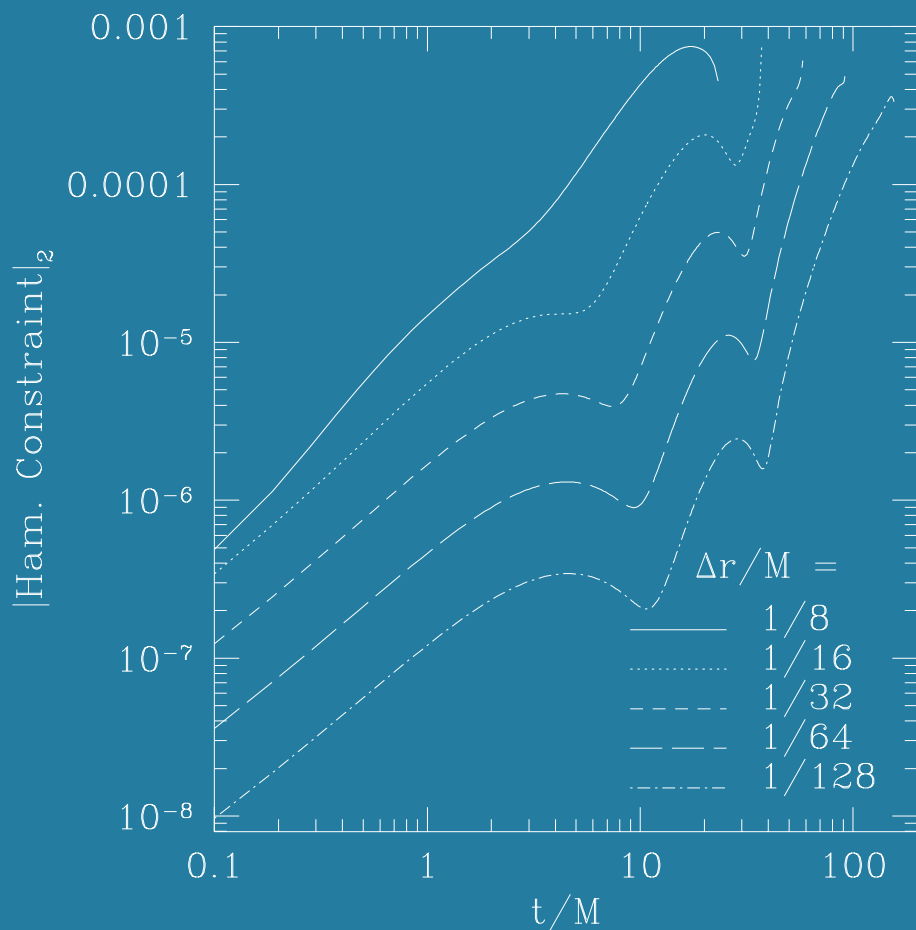
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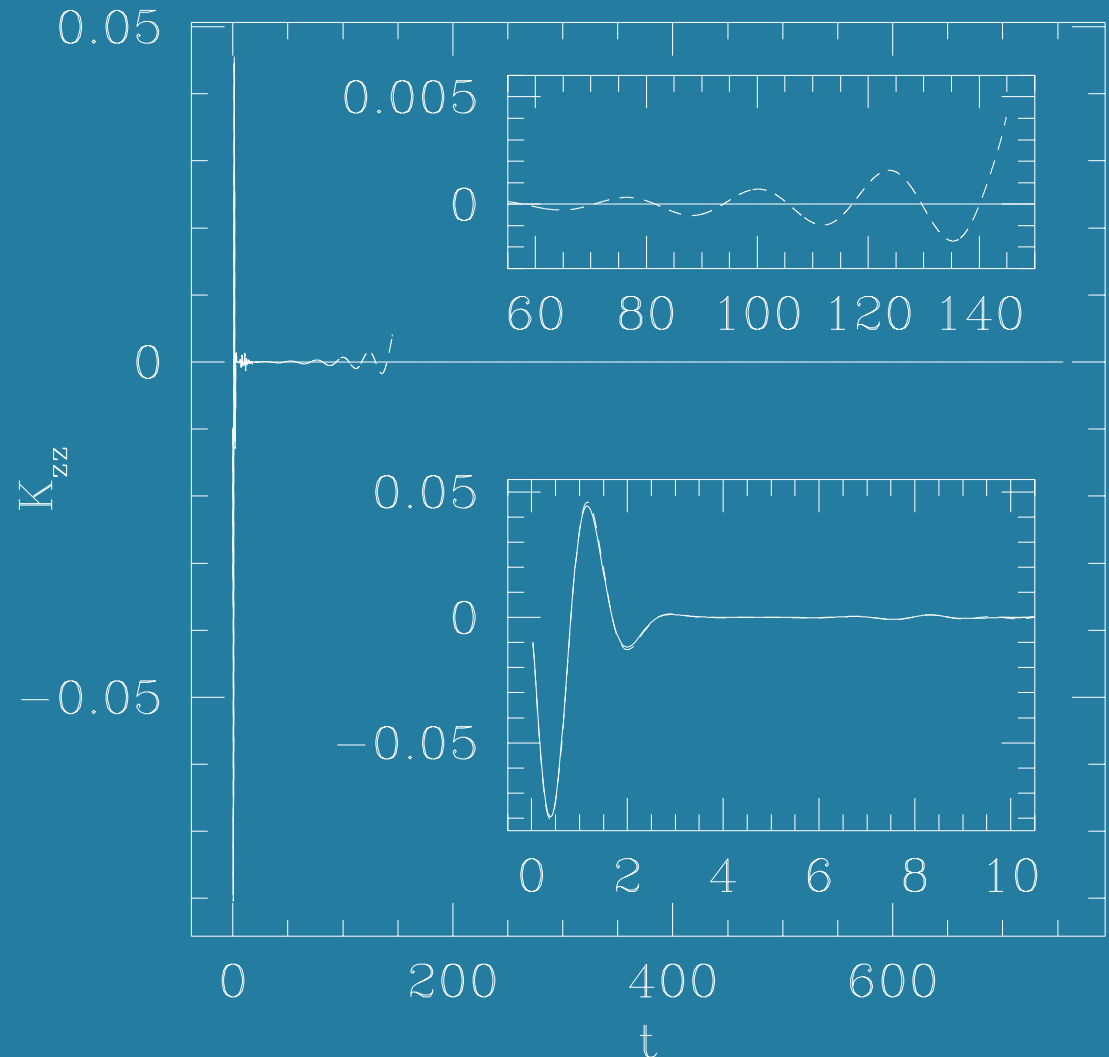
Formulation affects numerical results (1D)

- Spherically symmetric (1D) evolution of Schwarzschild BH.
- Add **algebraic** constraint to evolution eqns, numerics **unchanged**.



Formulation affects numerical results (GW) ⁷

- 3D weak gravitational waves (full nonlinear equations).
- Dashed: ADM
- Solid: “BSSN”



Boundary Conditions

- Three types of boundaries



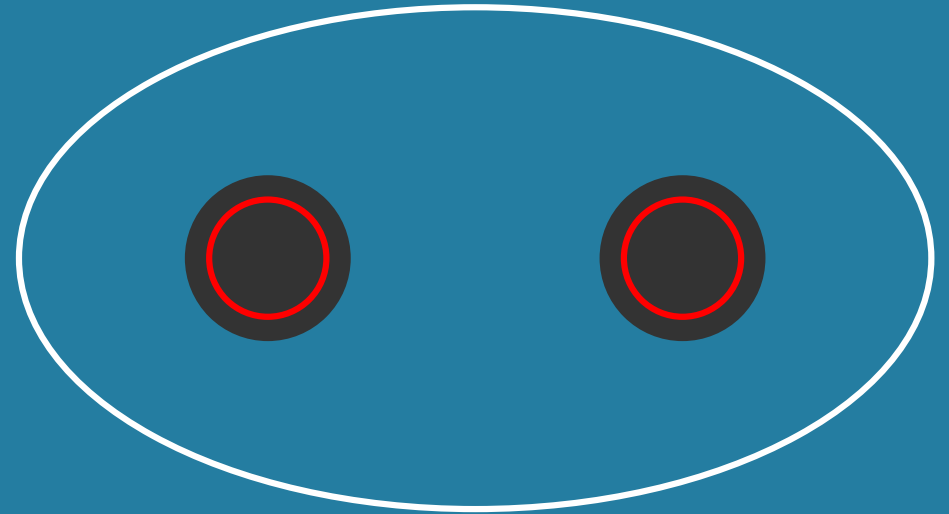
Boundary Conditions

- Three types of boundaries
 - ★ Excision



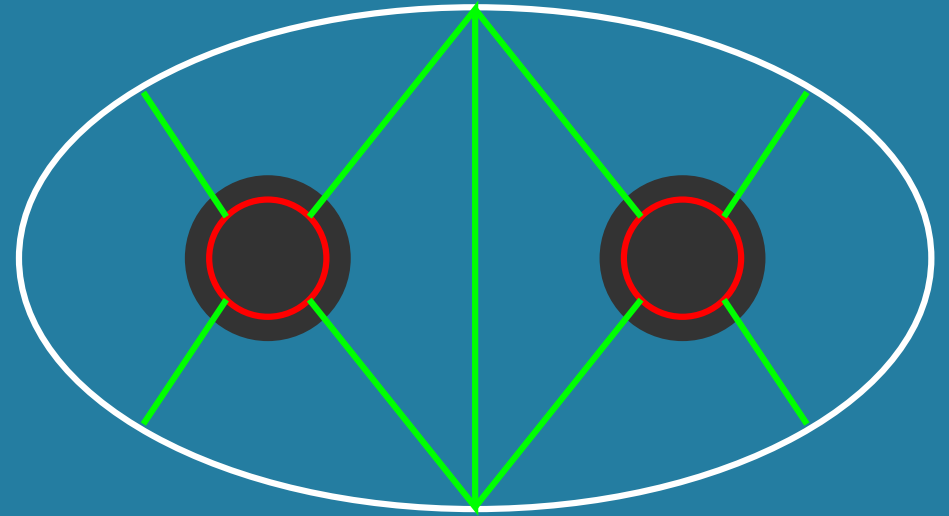
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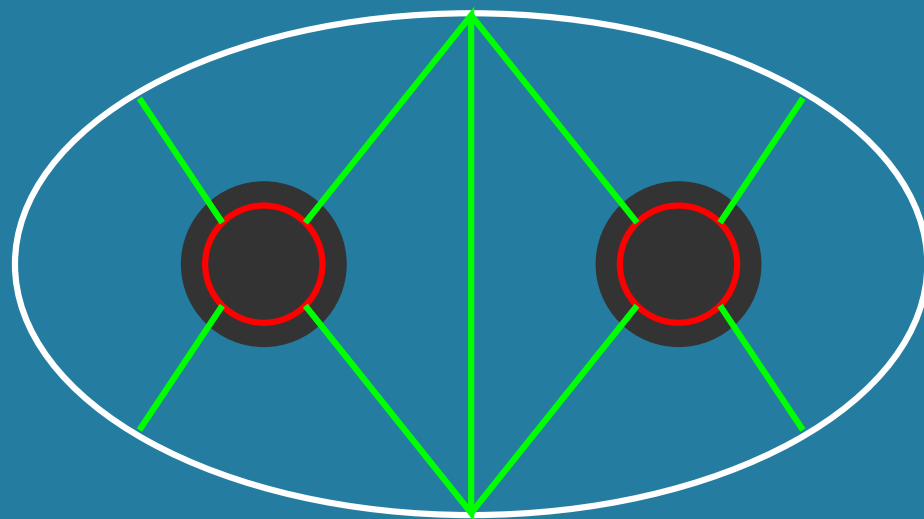
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Boundary Conditions

- Three types of boundaries

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- Q: What boundary conditions are required/allowed?

$$\begin{array}{c}
 Au_{,y} + Bu = C \\
 \left[\begin{array}{c} Au_{,x} + Bu = C \\ u_{,yy} + u_{,xx} = 0 \\ Au_{,x} + Bu = C \end{array} \right] \\
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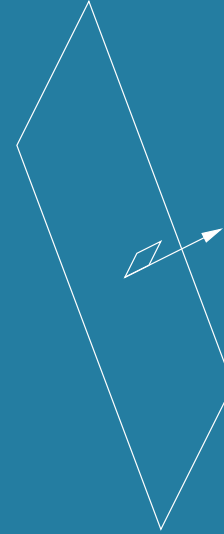
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Hyperbolicity

- Some formulations are **hyperbolic**.
 - ★ Hyperbolic: Complete set of *characteristic eigenfields*.
 - ★ Example: wave equation $\square\phi = 0$
 - ★ Example: $\partial_t u + C^i(u)\partial_i u = F(u)$
(if C^i are symmetric matrices).

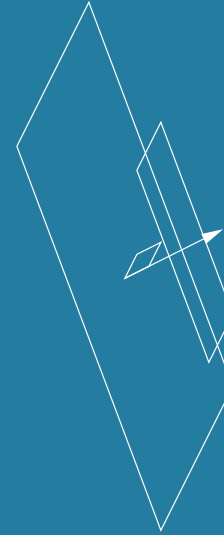
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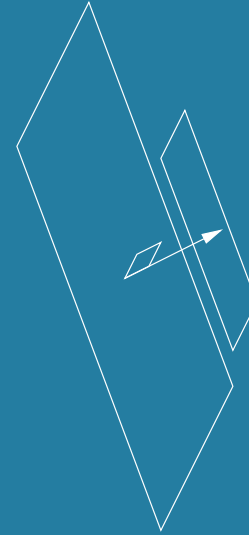
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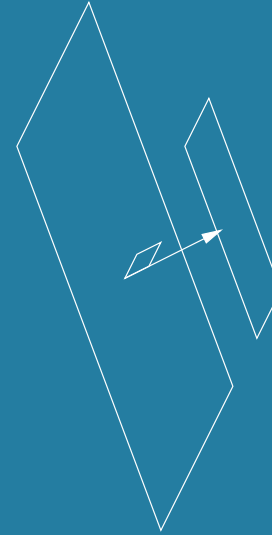
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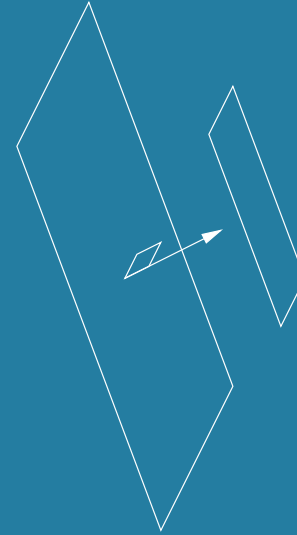
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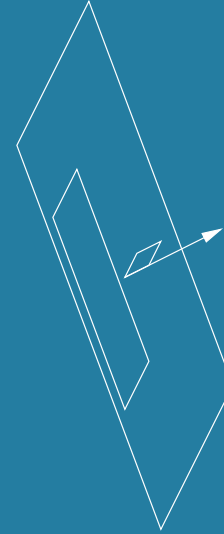
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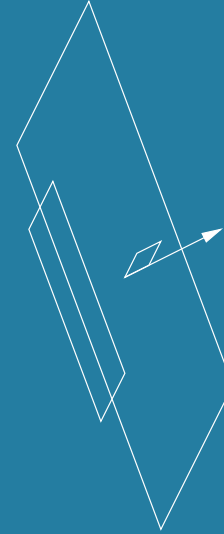
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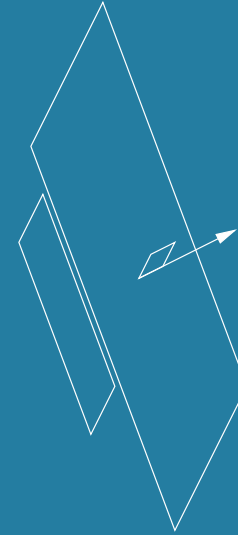
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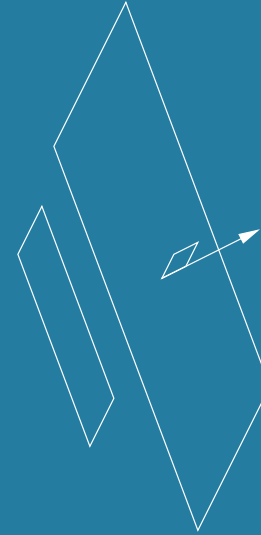
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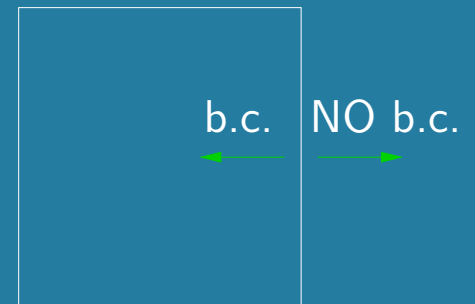
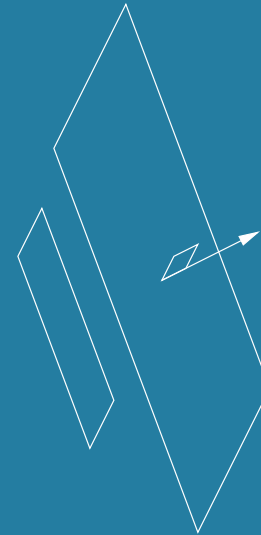
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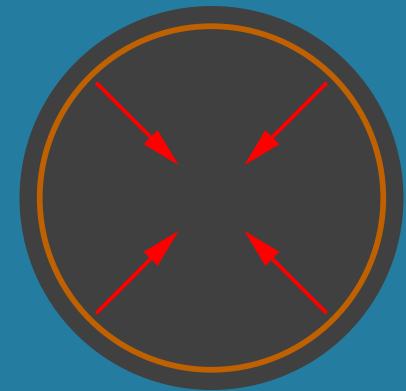
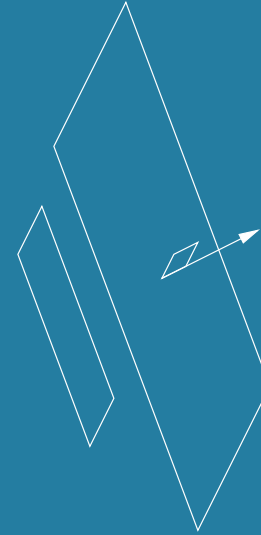
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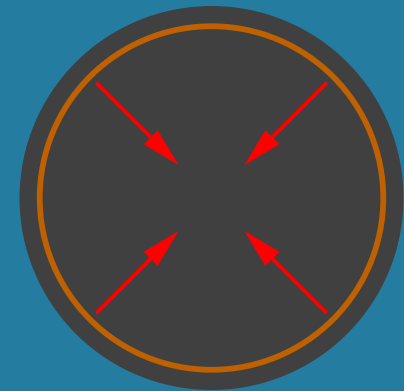
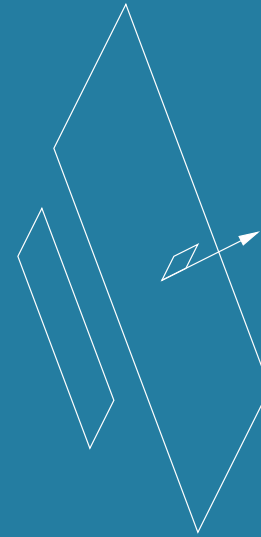
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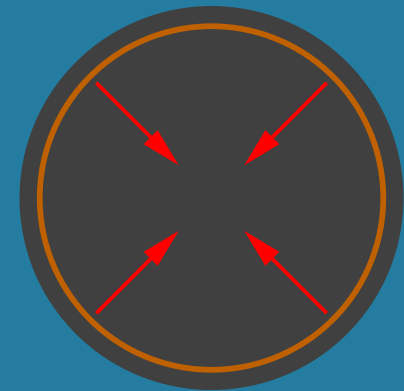
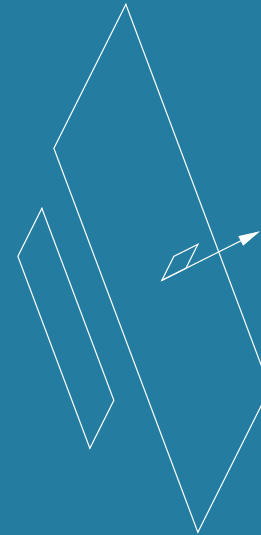
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- Some numerical methods require hyperbolicity.
- Usual 3+1 (ADM) formulation is **not** hyperbolic except in particular gauge.



Example of a hyperbolic formulation

“Einstein-Christoffel” (Anderson, York, Choquet-Bruhat 1999)

$$\begin{aligned} \partial_t g_{ij} - \beta^k \partial_k g_{ij} &= -2\alpha K_{ij} + 2g_{k(i} \partial_{j)} \beta^k, \\ \partial_t K_{ij} - \beta^k \partial_k K_{ij} + \alpha g^{k\ell} \partial_\ell f_{kij} &= \alpha X(g_{ij}, K_{ij}, f_{kij}), \\ \partial_t f_{kij} - \beta^\ell \partial_\ell f_{kij} + \alpha \partial_k K_{ij} &= \alpha Y(g_{ij}, K_{ij}, f_{kij}), \end{aligned}$$

where

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Compare scalar wave equation $\square\phi = 0$ in curved space with

$$\Pi \equiv \alpha^{-1}(\partial_t \phi - \beta^i \partial_i \phi), \quad \Phi_k \equiv \partial_k \phi:$$

$$\begin{aligned} \partial_t \phi - \beta^k \partial_k \phi &= -\alpha \Pi \\ \partial_t \Pi - \beta^k \partial_k \Pi + \alpha g^{k\ell} \partial_\ell \Phi_k &= \alpha X(\Pi, \Phi_k), \\ \partial_t \Phi_k - \beta^\ell \partial_\ell \Phi_k + \alpha \partial_k \Pi &= \alpha Y(\Pi, \Phi_k) \end{aligned}$$

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 - ★ Q: Can they be derived from an action principle?

Why not ADM in harmonic coordinates?

- Useful to retain gauge freedom
 - ★ Choose coordinate systems to simplify problem
 - * Time-indep coords for time-indep solutions
 - * Co-rotating coords for BBH inspiral
 - ★ Keep coordinates nonsingular.
- Harmonic coords can be ill-behaved
 - ★ Harmonic foliation of flat space can develop coordinate singularities.
 - ★ Harmonic foliations of some time-independent space-times are time-dependent.