Numerical Computation of Black Hole Spacetimes

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Caltech

Outline:

- Introduction
- How are Einstein's equations solved?
- Why is it so difficult to evolve black holes?

Goal

Solve Einstein's equations for binary black hole system

- Dynamical time scale for a single BH is its mass M.
  (Reminder: $G = c = 1.$ $\qquad M_\odot = 5\mu\text{s} = 1.5\text{km}$)
- Must follow evolution for thousands of M.

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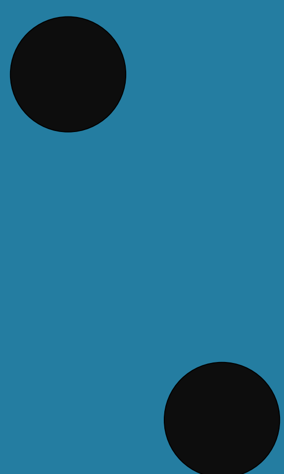

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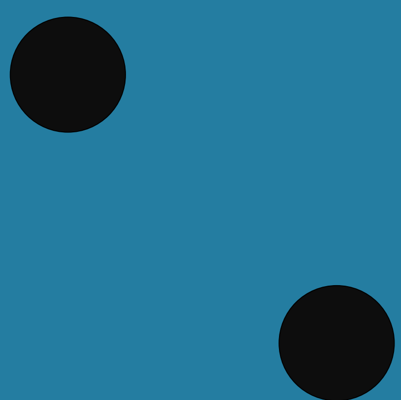

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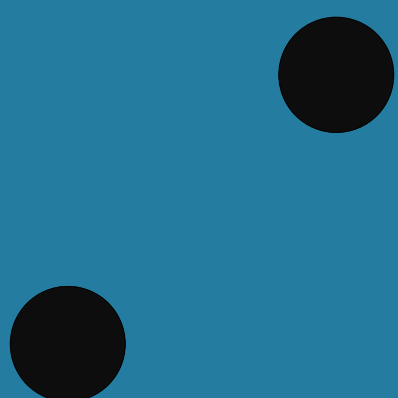

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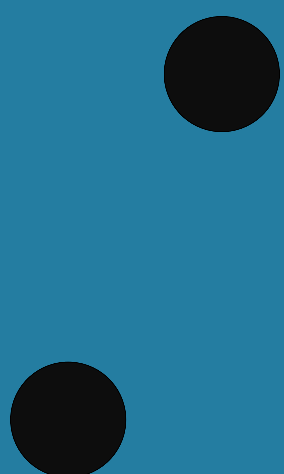

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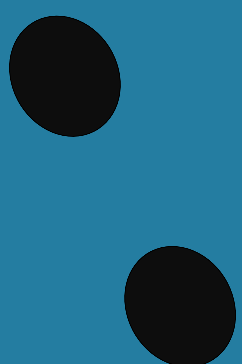

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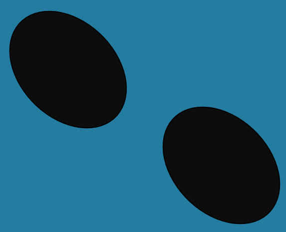

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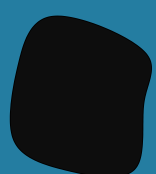

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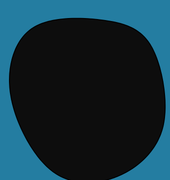

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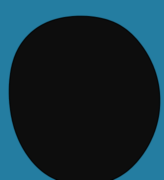

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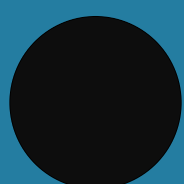

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- All current general methods are unstable!
  - ⋆ Codes crash at $t \sim 30M$—$1000M$

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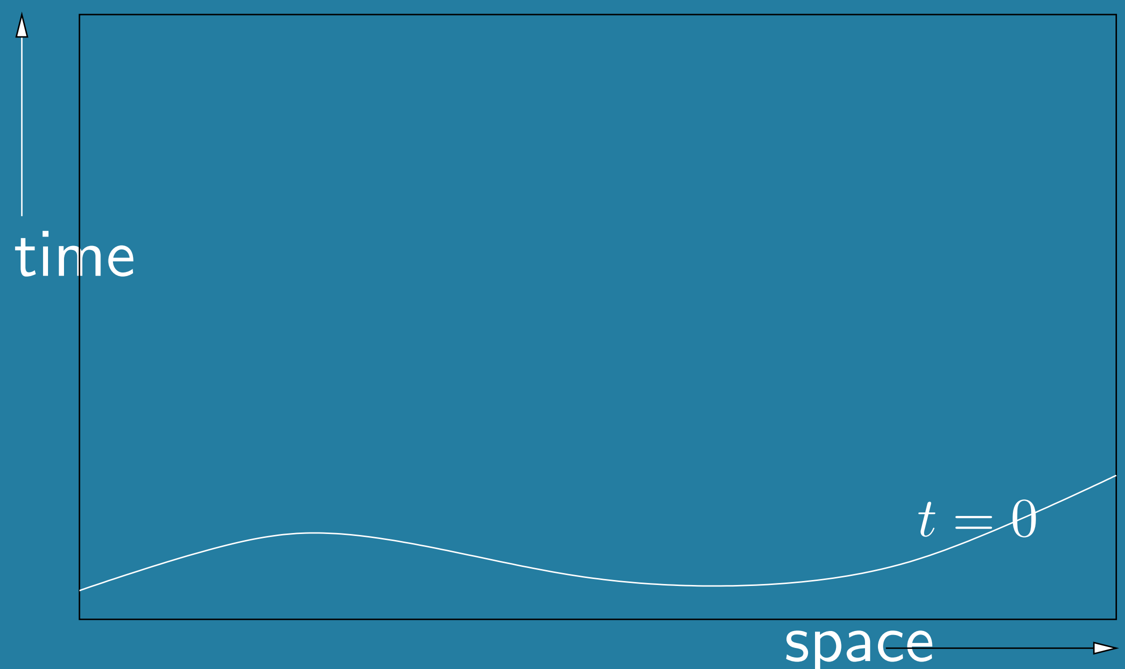

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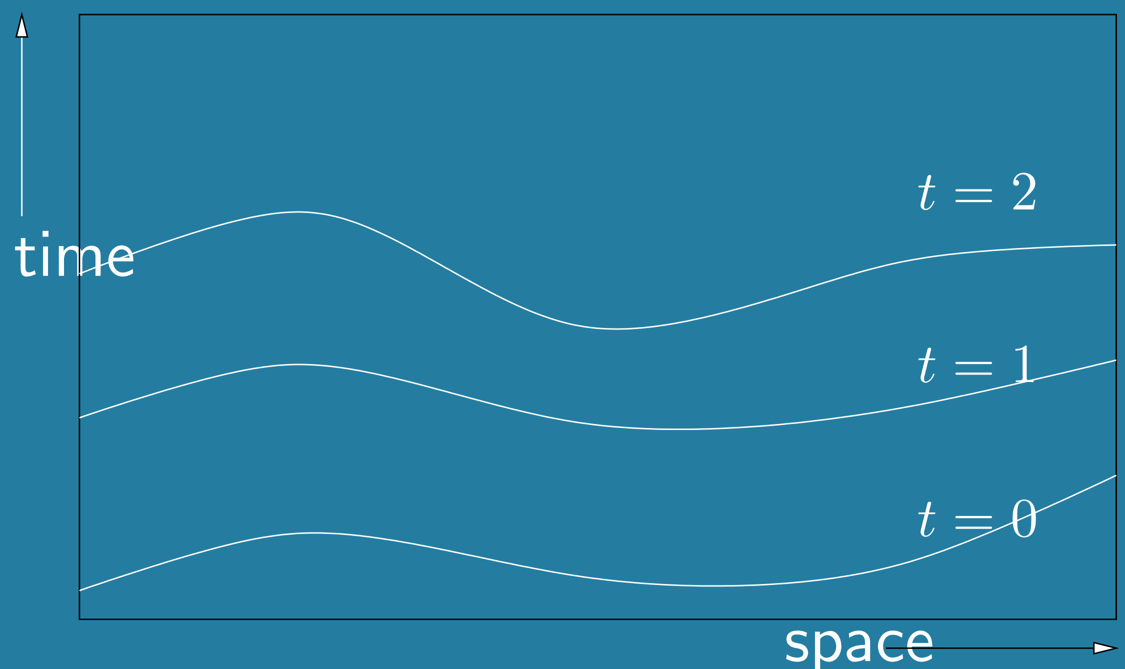

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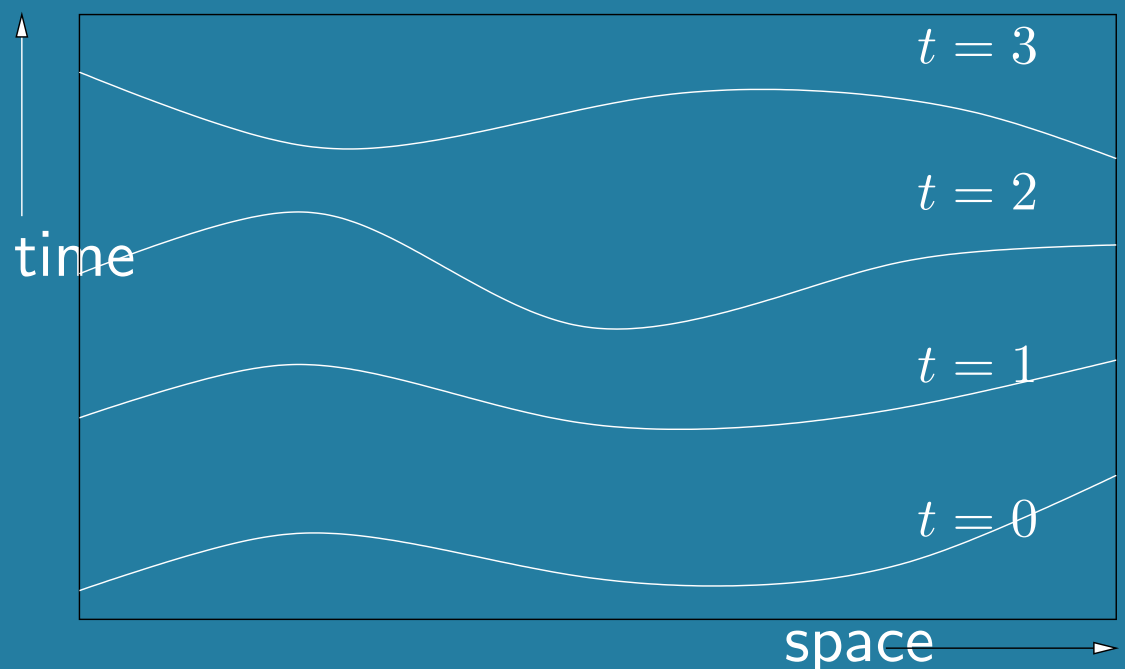

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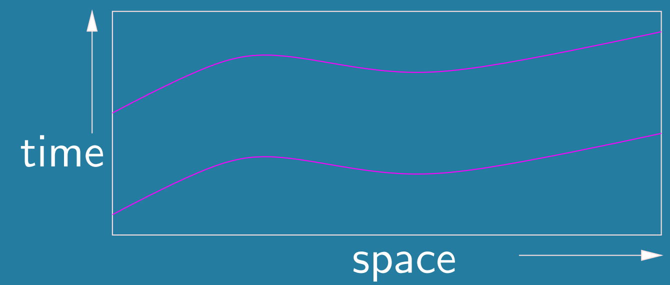

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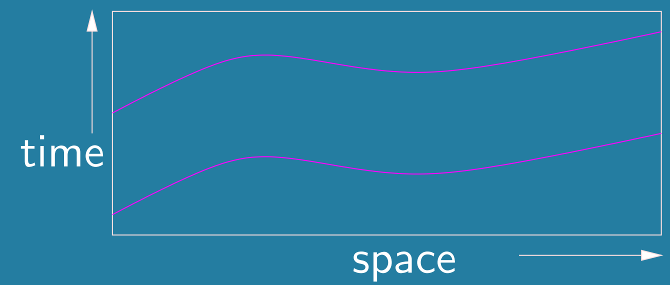

- Then
$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= 4\pi\rho && (C1) \\ \partial_t \vec{A} &= -\vec{E} - \vec{\nabla}\phi && (E1) \\ \partial_t \vec{E} &= \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) - 4\pi\vec{J} && (E2) \end{aligned}$$

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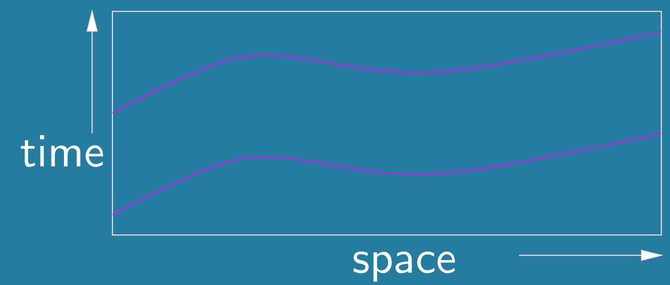

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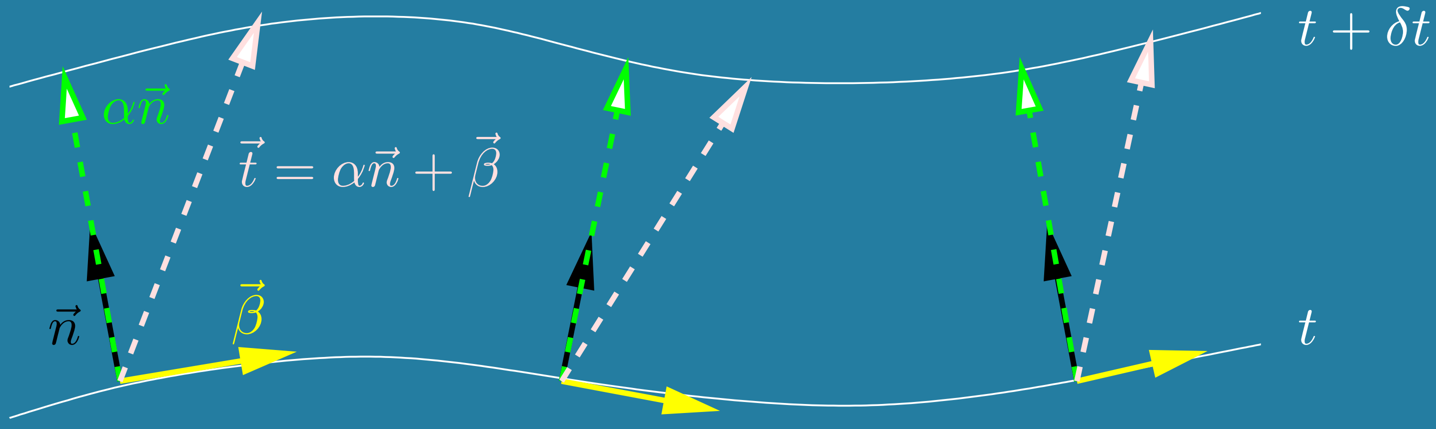
- Scalar potential ϕ undetermined (gauge).
- Vector potential $\vec{A}$ still has some gauge freedom.

3+1 Variables for General Relativity

$$ds^2 = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i\, dt)(dx^j + \beta^j\, dt)$$



- Three-metric g_{ij}: Measures lengths at constant t. (Analogue: $\vec{A}$)
- Lapse α: Measures proper time of normal observer. (Analogue: ϕ)
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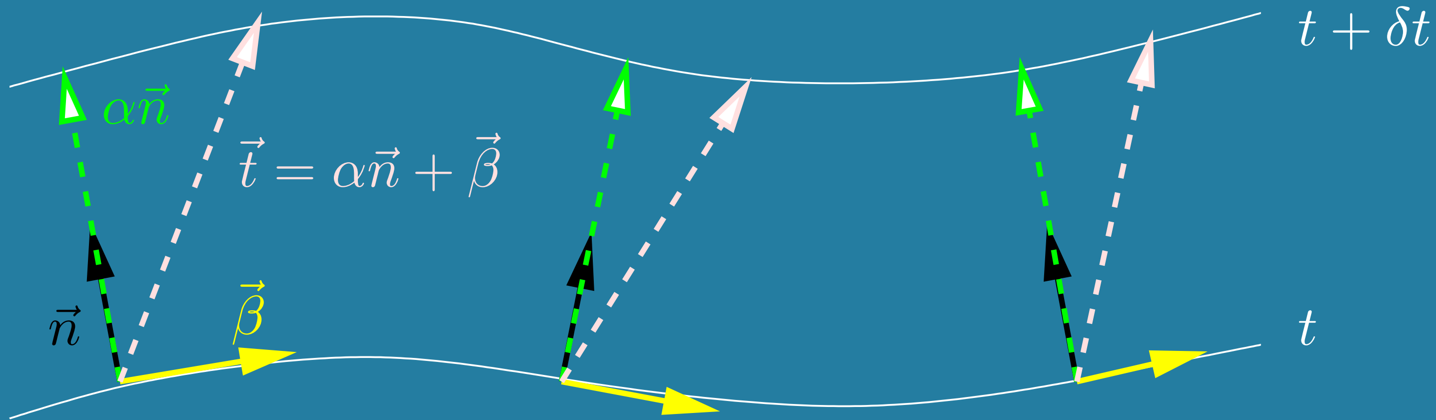

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- Extrinsic Curvature $K_{ij} \equiv -\frac{1}{2}\pounds_{\vec{n}} g_{ij}$
  Measures how normals diverge. (Analogue: $\vec{E}$)

3+1 Einstein Equations (ADM)

Constraint Equations:

$$\begin{aligned} D^j K_{ij} - D_i K^j{}_j &= 8\pi S_i \\ \bar{R}^i{}_i + K^i{}_i K^i{}_i - K_{ij} K^{ij} &= 16\pi\rho \end{aligned}$$

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Evolution Equations:

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Maxwell–Einstein Analogy

	Maxwell	Einstein
Variables	A_μ	$g_{\mu\nu}$
Equation	$\partial^\nu F_{\mu\nu} = 4\pi J_\mu$	$G_{\mu\nu} = 8\pi T_{\mu\nu}$
3+1	$A_\mu \to \vec{A}, \phi$	$g_{\mu\nu} \to g_{ij}, \alpha, \beta^i$
∂_t variables	$\vec{E}$	K_{ij}
Constraints	$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$	$F(K, g, K', g', g'') = 0$
Evolution	$\partial_t \vec{A}, \partial_t \vec{E}$	$\partial_t g_{ij}, \partial_t K_{ij}$
Gauge[1]	$A_0 \equiv \phi$	$g_{0\mu} \equiv \alpha, \beta^i$
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	Linear	Nonlinear

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3+1 Equations Overdetermined

Maxwell Illustration:

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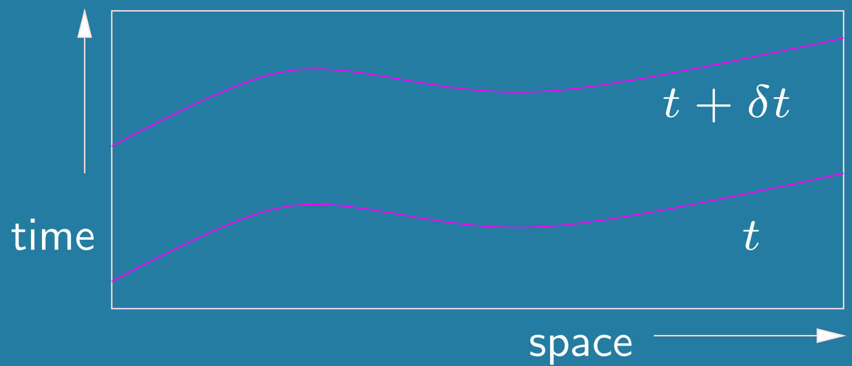

- Given $\vec{E}$, $\vec{A}$ at t, how does one get $\vec{E}$, $\vec{A}$ at $t + \delta t$?
- Usually choose free evolution.

Why so difficult?

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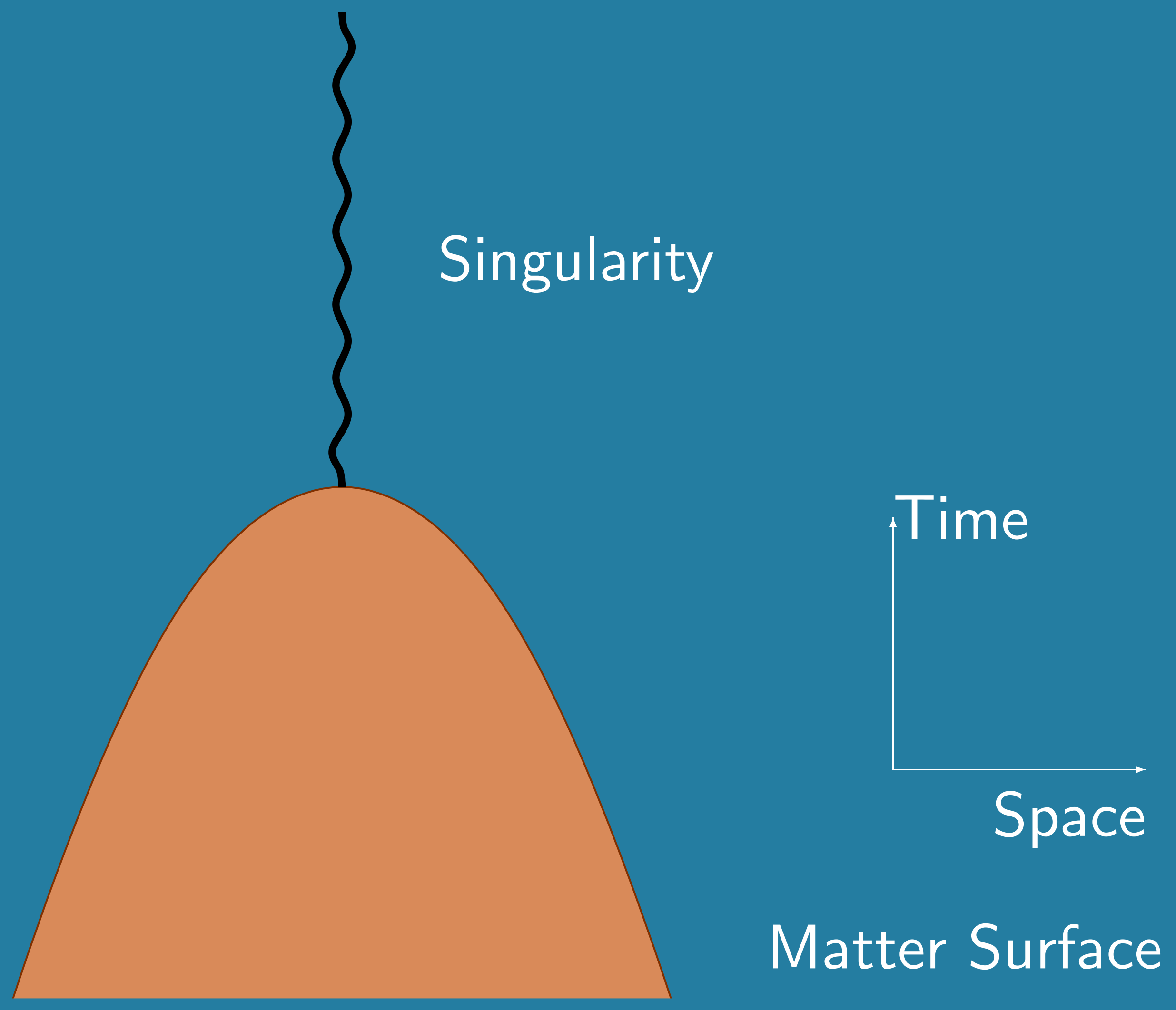
Bigger computers will not solve our problems!

Anatomy of a Black Hole



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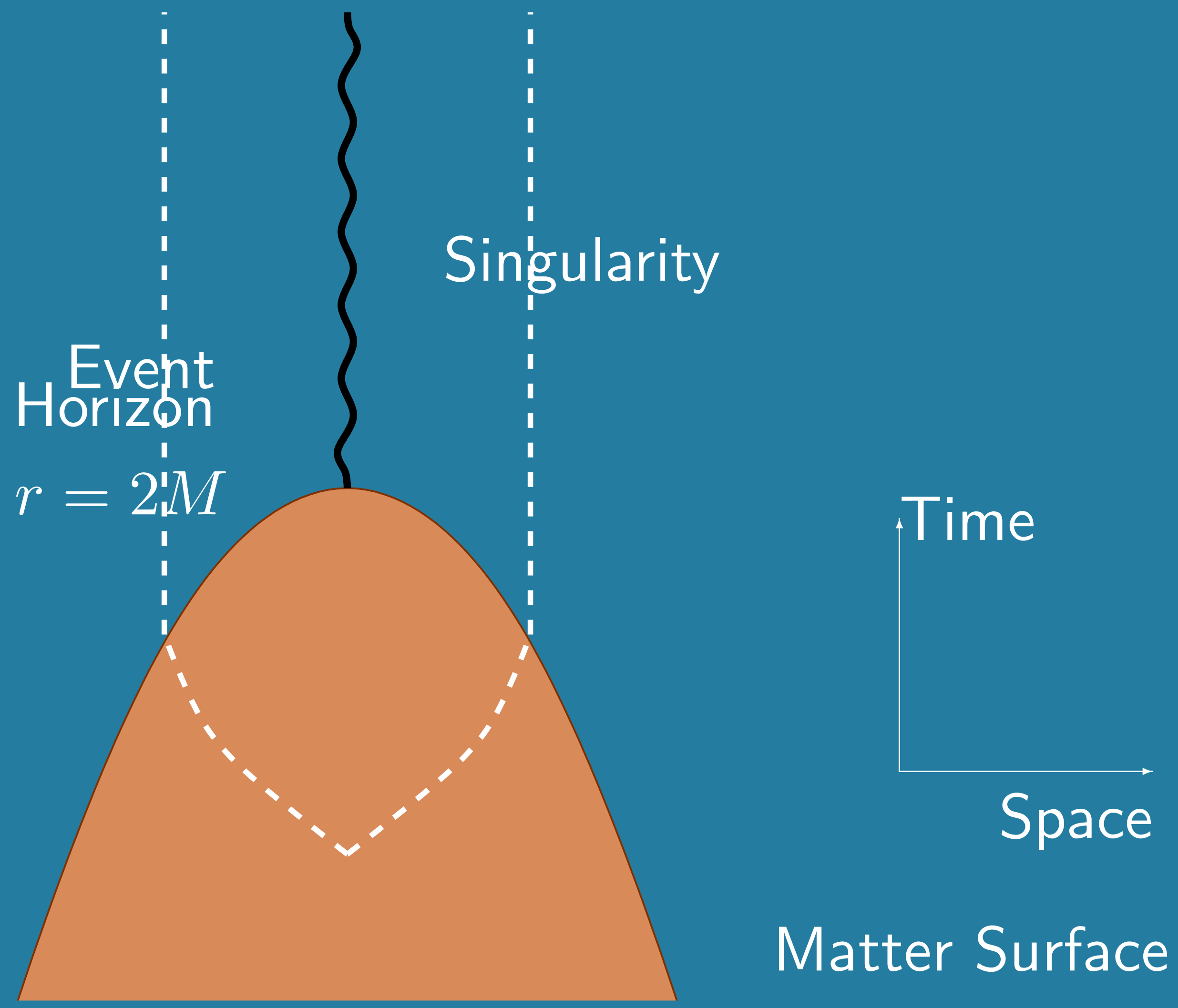

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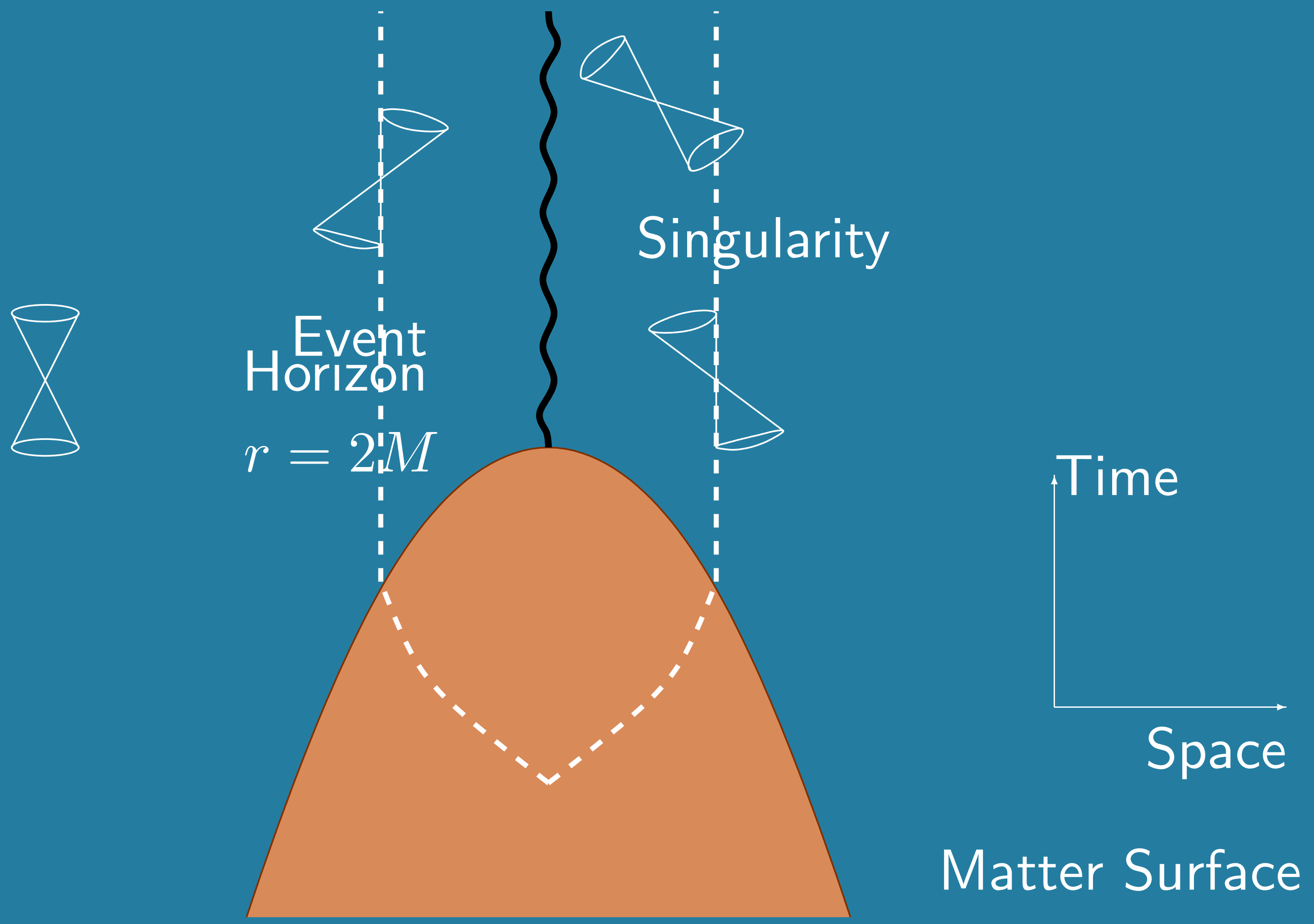

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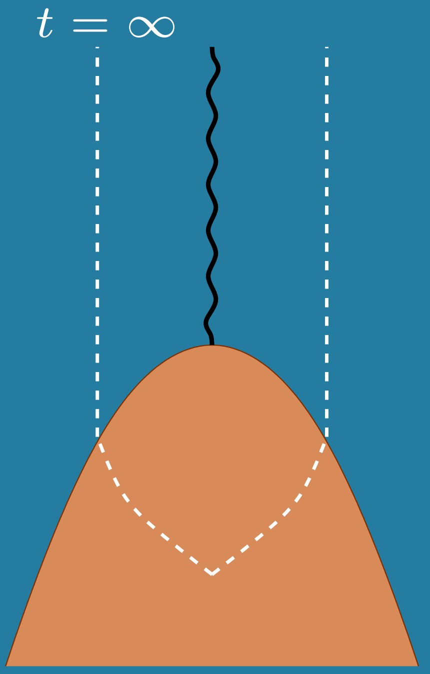
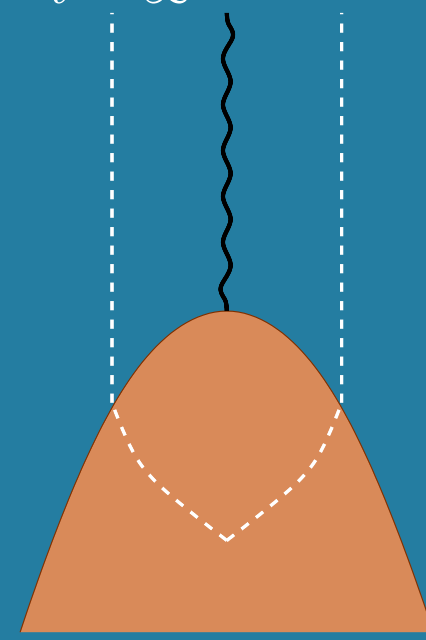


- Hard to find good coordinate system
  - ⋆ Many bad ones (e.g. Schwarzschild)

$$ds^2 = -(1 - 2M/r)\, dt^2 + (1 - 2M/r)^{-1}\, dr^2 + r^2\, d\Omega^2$$

Coordinate invariance



- Hard to find good coordinate system
  - ⋆ Many bad ones (e.g. Schwarzschild)

  $ds^2 = -(1 - 2M/r)\, dt^2 + (1 - 2M/r)^{-1}\, dr^2 + r^2\, d\Omega^2$
- Some solutions of Einstein's equations correspond to coordinate motion.

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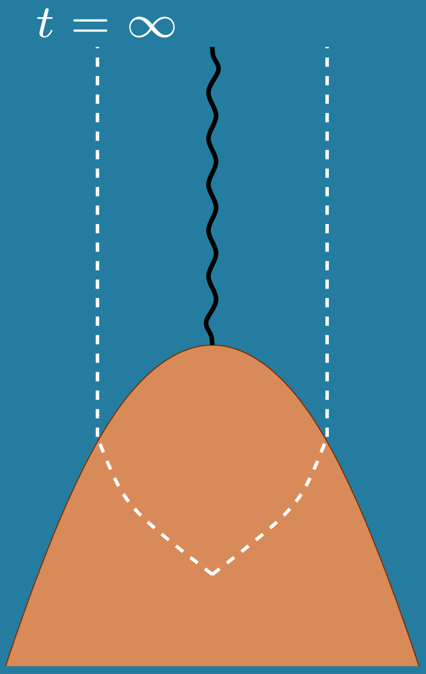

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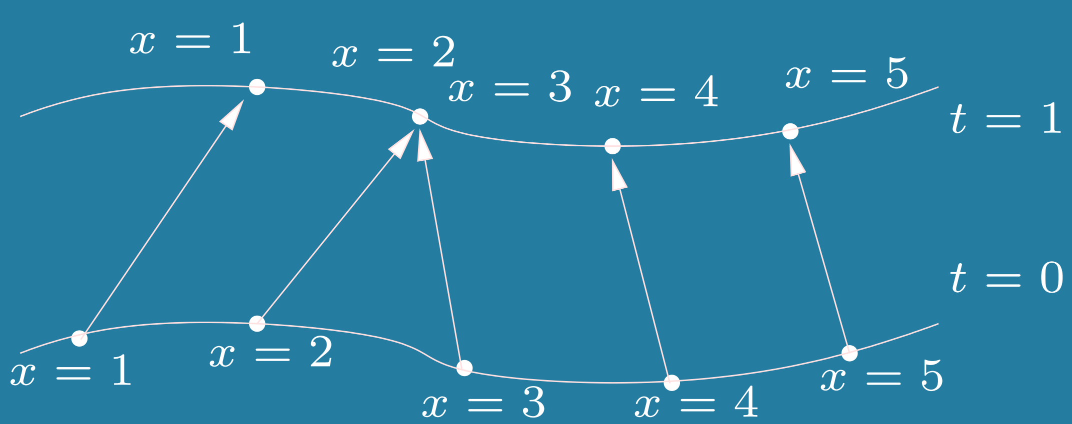
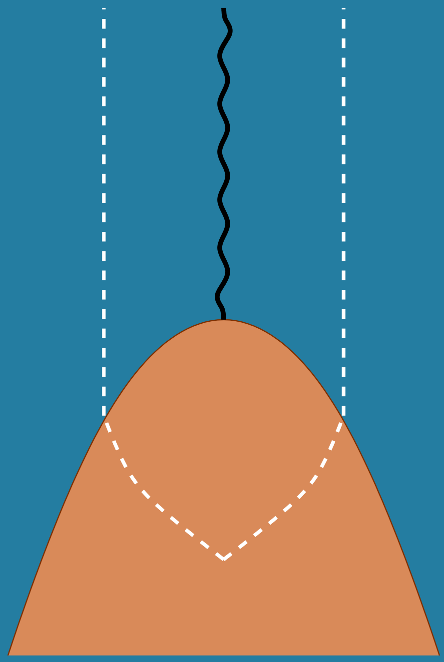
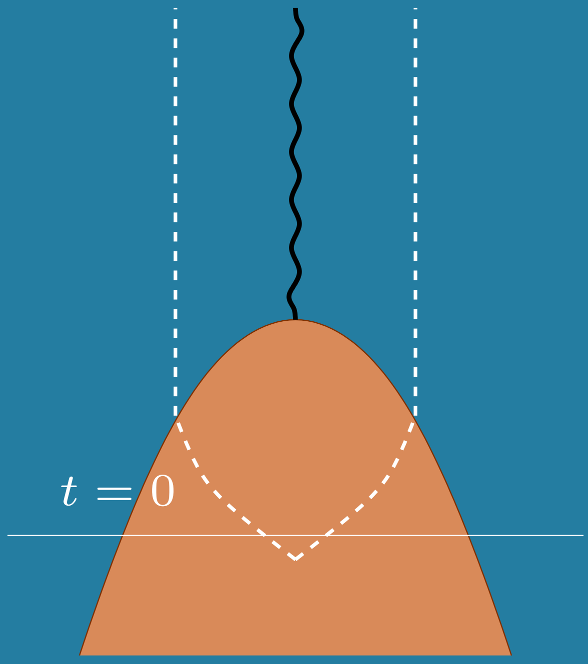


Singularities



Singularities

- Naive integration hits singularity



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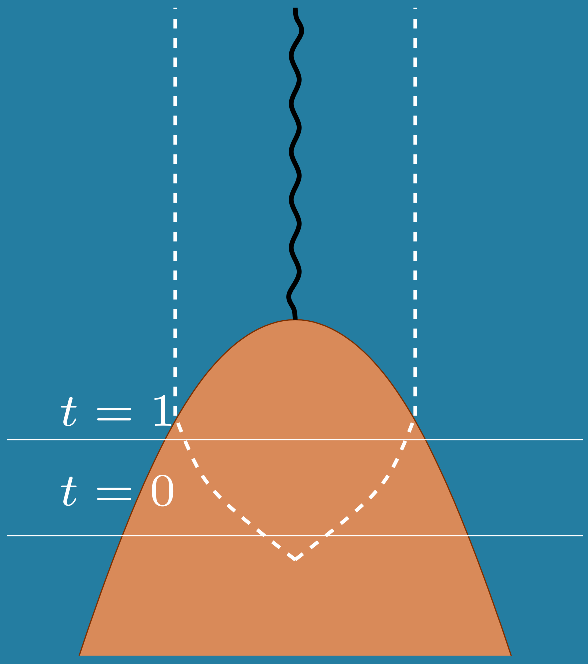

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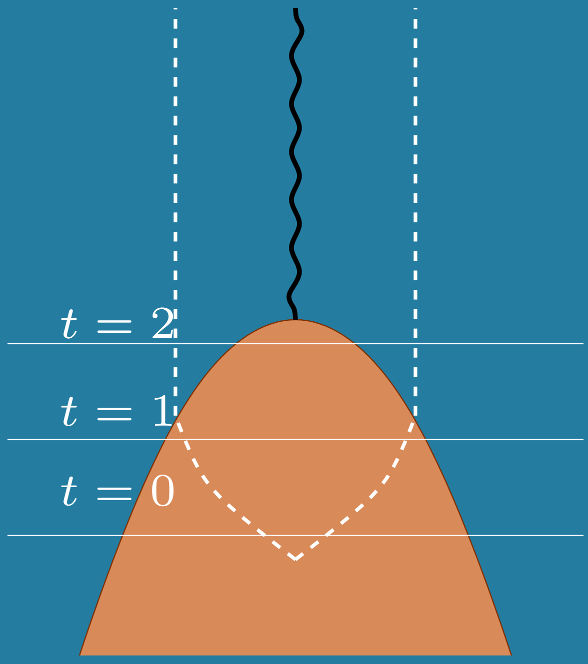

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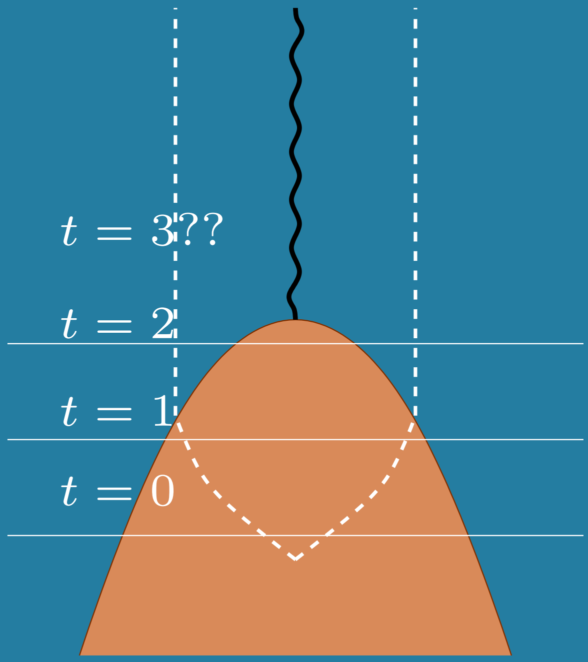

Singularities

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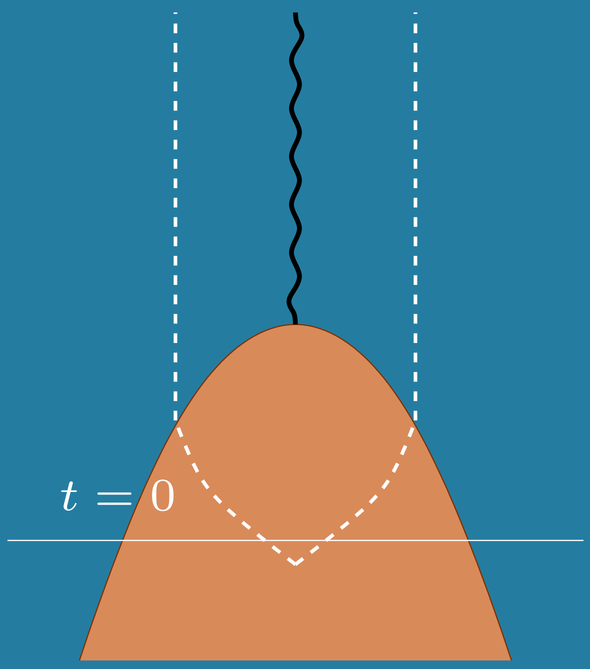

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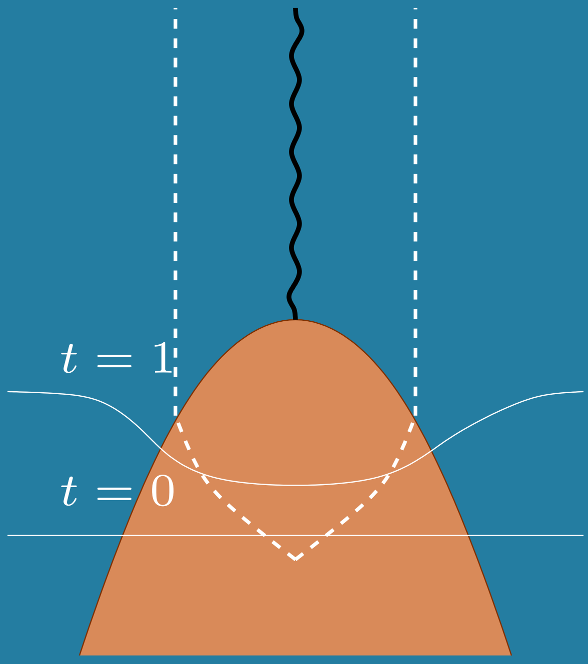

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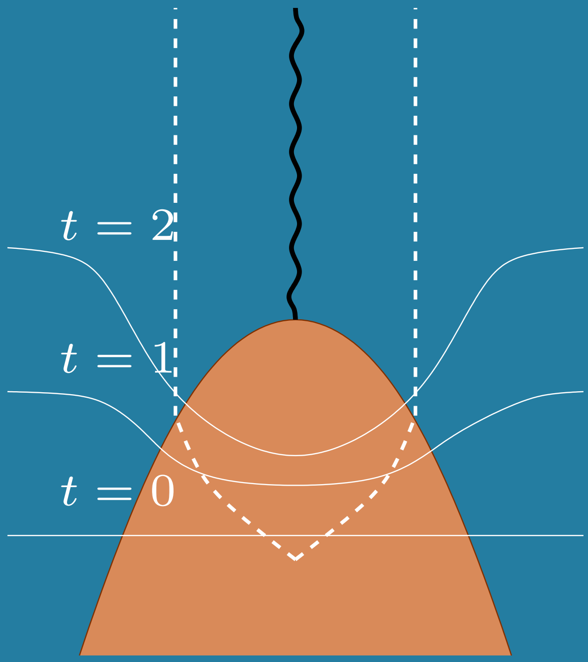

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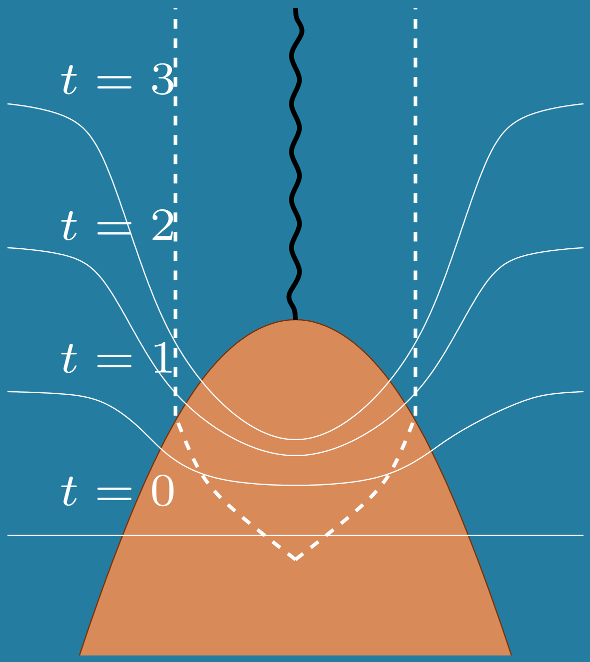

Singularities

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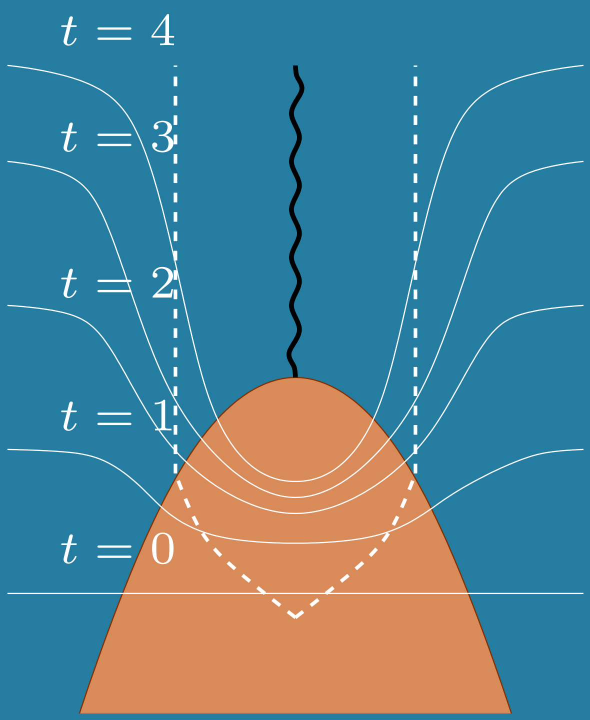
- Try coordinate freedom
  - ⋆ Grid stretching



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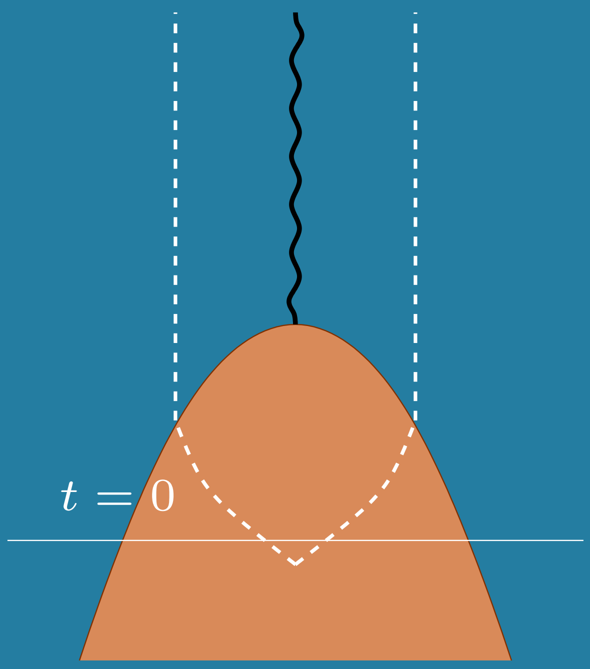
- Modern method: Excision



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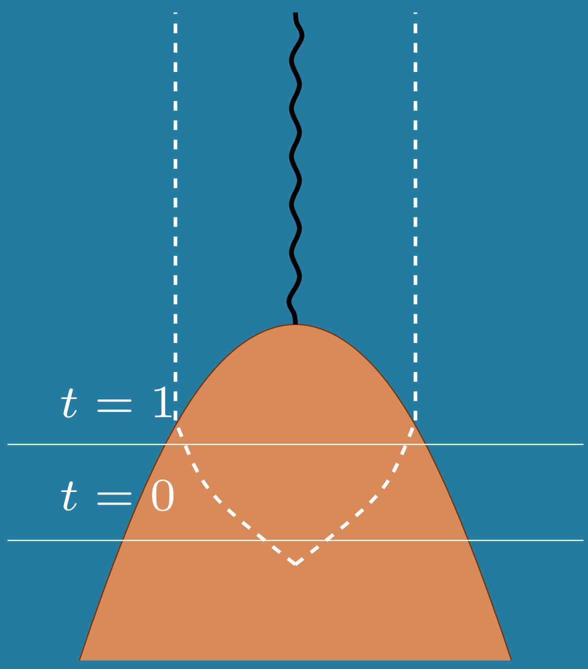

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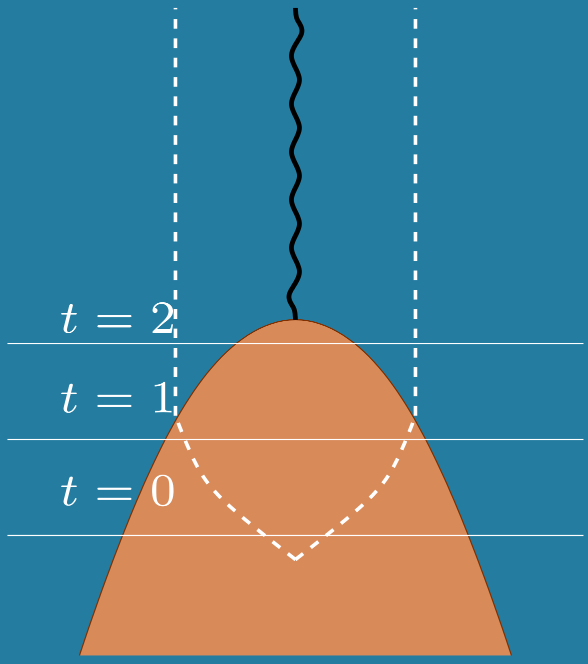

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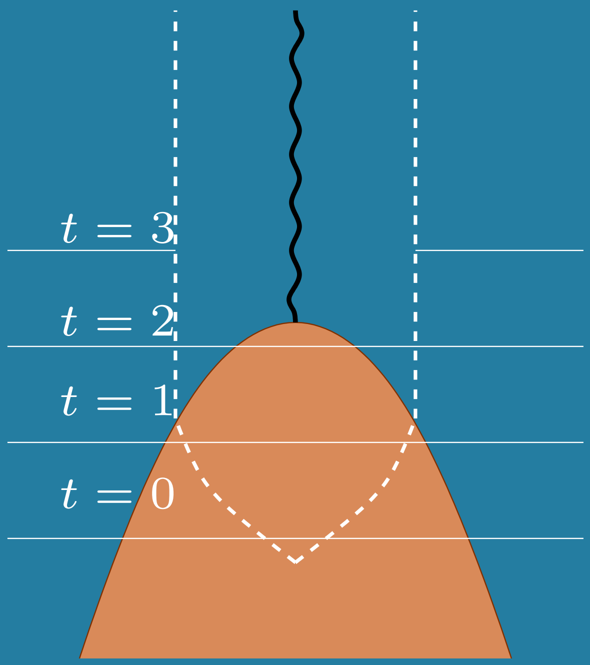

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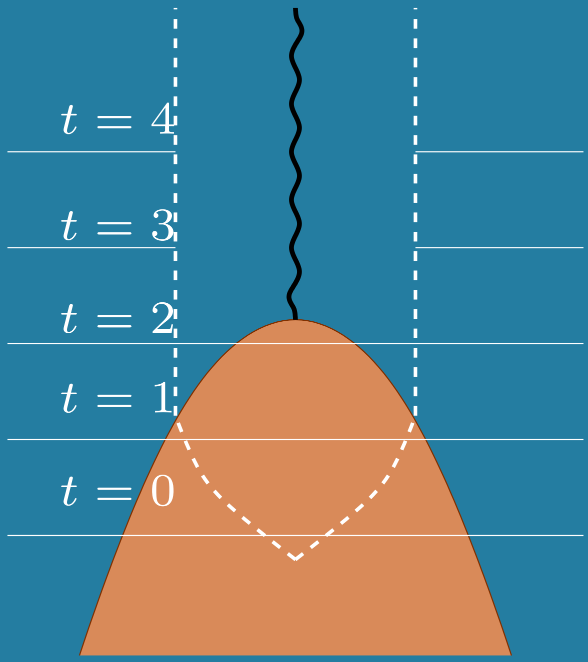
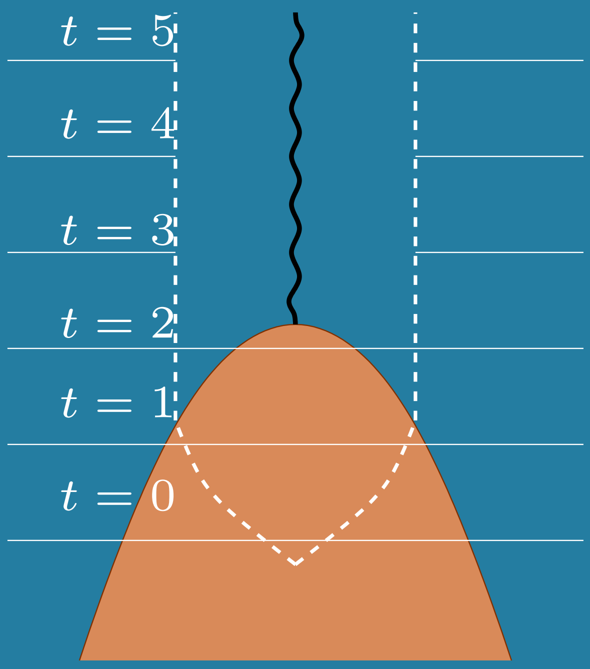


Singularities

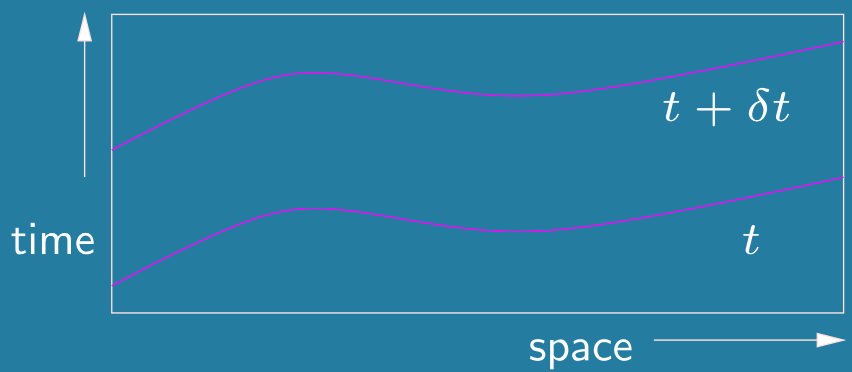
- Naive integration hits singularity
- Try coordinate freedom
  - ⋆ Grid stretching
- Modern method: Excision
  - ⋆ Boundary conditions?
  - ⋆ Horizon changes shape.



Constraint Violation

Maxwell Illustration:

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= 4\pi\rho && (C1)\\
\partial_t \vec{A} &= -\vec{E} - \vec{\nabla}\phi && (E1)\\
\partial_t \vec{E} &= \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) - 4\pi\vec{J} && (E2)
\end{aligned}
$$



- Does solving (E1), (E2) guarantee (C1)?
  - ⋆ Exact solution: yes.
  - ⋆ Approximate solution: no.

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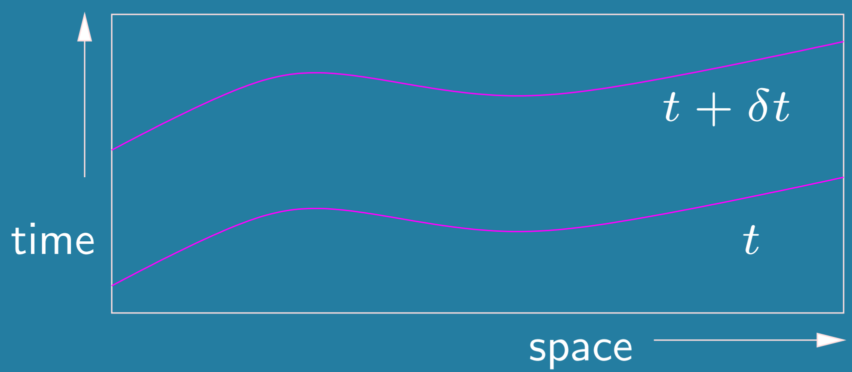

- Does solving (E1), (E2) guarantee (C1)?
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  - ⋆ Problem: 3+1 Einstein evolution equations:
    Perturbations that violate constraints can grow rapidly.

Definitions

$$D_i \equiv \partial_i + \text{terms like } g_{jk,i}$$

$$\begin{aligned}
\bar{R}_{ij} \;\; &\equiv \;\; g^{pq} g^{rs} \left(\frac{1}{2} g_{rs,p} g_{q(i,j)} - g_{qs,r} g_{p(i,j)} + \frac{1}{4} g_{ps,j} g_{qr,i} \right. \\
&\qquad \left. + g_{pj,s} g_{i[q,r]} + \frac{1}{4} g_{qs,r} g_{ij,p} - \frac{1}{4} g_{rs,q} g_{ij,p} \right) \\
&+ \;\; g^{pq} \left(g_{p(i,j)q} - \frac{1}{2} g_{ij,pq} - \frac{1}{2} g_{pq,ij} \right)
\end{aligned}$$