Introduction to General Relativity

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- General relativity is Einstein's theory of gravitation.
- Newtonian concept of gravitational force is replaced by curved spacetime.
- Test particles move along geodesics of spacetime.
- Spacetime curvature is determined by density and pressure of matter.

• Geometry of spacetime is described by the metric tensor g_{ab}.

• For flat empty spacetime $g_{ab} = \eta_{ab} = \mathrm{diag}(-1,1,1,1)$ in Cartesian coordinates,

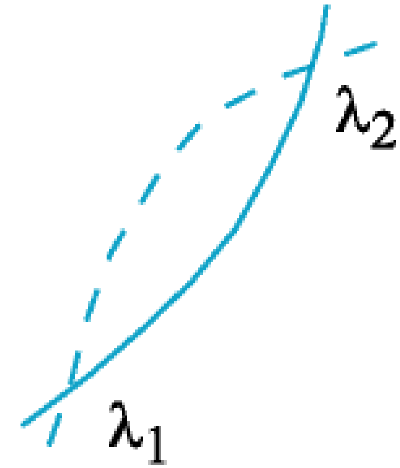
$$ds^2 = g_{ab}dx^a dx^b = -dt^2 + dx^2 + dy^2 + dz^2.$$

• The metric determines the causal structure of spacetime:

$$\text{If} \left\{ \begin{array}{c} g_{ab}u^a u^b < 0 \\ g_{ab}u^a u^b = 0 \\ g_{ab}u^a u^b > 0 \end{array} \right\} \text{then } u^a \text{ is} \left\{ \begin{array}{c} \text{timelike} \\ \text{null} \\ \text{spacelike} \end{array} \right\}.$$

- Let $x^a(\lambda)$ denote a curve in spacetime.
- The proper distance between two points along the curve is

$$D = \int_{\lambda_1}^{\lambda_2} \sqrt{\left| g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} \right|} d\lambda$$



- The curve that extremizes the distance between points satisfies the equation:

$$\frac{d^2 x^a}{d\lambda^2} + \Gamma^a_{bc} \frac{dx^b}{d\lambda} \frac{dx^c}{d\lambda} = 0,$$

where $\Gamma^a_{bc} = \frac{1}{2} g^{ae} (\partial_b g_{ec} + \partial_c g_{eb} - \partial_e g_{bc})$.

• The curvature of spacetime is described by several tensors (complicated nonlinear functions of the spacetime metric):

$$
\begin{aligned}
R^e{}_{cab} &= \partial_a \Gamma^e_{bc} - \partial_b \Gamma^e_{ac} \\
&\quad + \Gamma^d_{bc}\Gamma^e_{ad} - \Gamma^d_{ac}\Gamma^e_{bd} && \text{Riemann} \\
R_{ab} &= R^e{}_{aeb} && \text{Ricci} \\
R &= g^{ab} R_{ab} && \text{Scalar}
\end{aligned}
$$

where

$$\Gamma^a_{bc} = \frac{1}{2} g^{ae}(\partial_b g_{ec} + \partial_c g_{eb} - \partial_e g_{bc})$$

and g^{ae} is the inverse of g_{eb}: $g^{ae} g_{eb} = \delta^a{}_b$.

- In general relativity theory the Ricci curvature is determined by the mass-energy density and pressure of matter.

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}$$

This is called Einstein's equation.

- T_{ab} is the stress-energy tensor of matter.

T_{tt} — mass-energy density

T_{ti} — momentum density

T_{ij} — pressure and stress

where $i, j = 1, 2, 3$.

- In the vacuum region outside stars (or black holes), Einstein's equation becomes

$$R_{ab} - \frac{1}{2} g_{ab} R = 0$$

- The vacuum equations require that the Ricci curvature vanishes, but this does not imply that the spacetime is flat there.
- A "typical" component of the Riemann tensor in the exterior of a star (or black hole) is comparable to M/r^3.

- Einstein's equation can be derived from a variational principle:

$$S = \int R\sqrt{-g}d^4x,$$

where $g = \det g_{ab}$.

- Consider the one-parameter family of metrics $g_{ab}(\lambda)$, then

$$\frac{dS}{d\lambda} = -\int \left[(R^{ab} - \frac{1}{2}g^{ab}R)\delta g_{ab} + \nabla_a \delta v^a\right] \sqrt{-g}d^4x$$

where $\delta g_{ab} = \frac{dg_{ab}}{d\lambda}$,

and $\delta v^a = g^{bc}\nabla^a \delta g_{ab} - g^{ac}\nabla^b \delta g_{bc}$.

- One simple vacuum solution to Einstein's equation is the Schwarzschild geometry:

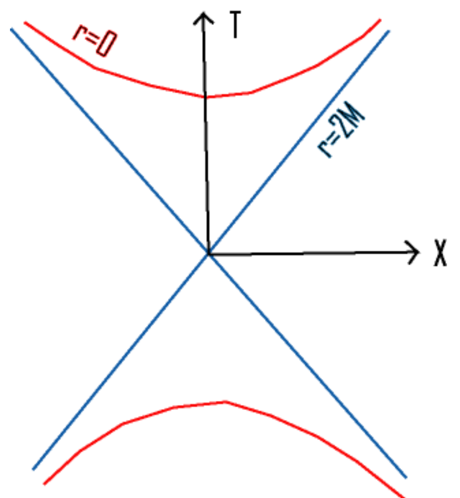
$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

- Define new coordinates for Schwarzschild

$$X^2 - T^2 = M^2\left(\frac{r}{2M} - 1\right) e^{r/2M}, \qquad \frac{T}{X} = \tanh\frac{t}{4M}$$

$$ds^2 = \frac{32M}{r} e^{-r/2M}\left(-dT^2 + dX^2\right) + r^2 d\Omega^2$$

- The Schwarzschild geometry is a black hole with horizon at $r = 2M$ and singularity at $r = 0$.

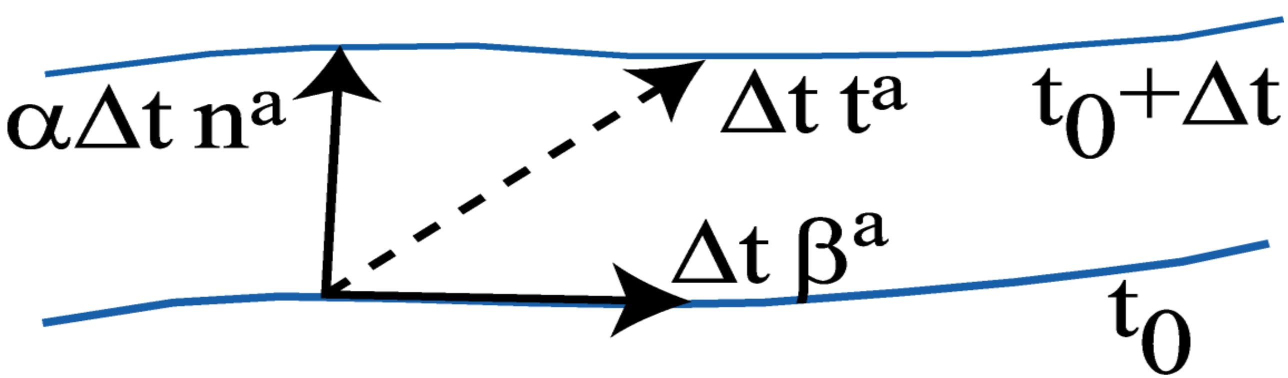


- Gravitational radiation is produced by moving masses in general relativity.
- For slowly moving sources with weak gravitational fields the spacetime metric that describes the gravitational waves is given (to lowest order) by

$$g^{ij} = \delta^{ij} + \frac{2}{r}\frac{d^2}{dt^2}\int T_{tt}(t-r,\vec{x})(x^i x^j - \frac{1}{3}r^2\delta^{ij})d^3x.$$

- The Cauchy problem consists of finding a spacetime from appropriate initial data specified on a spacelike surface.
- Consider a foliation of $t =$ constant spacelike surfaces, with spatial coordinates x^i defined on each surface. It is convenient to write the spacetime metric as:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$



• Einstein's (vacuum) equation can be decomposed into a set of evolution equations for the spatial metric γ_{ij} and the extrinsic curvature K_{ij},

$$\begin{aligned} \partial_t \gamma_{ij} &= -2\alpha K_{ij} + \ldots \\ \partial_t K_{ij} &= \alpha \left[R^{(3)}_{ij} + K_{ij} K - 2 K_{im} K^m{}_j \right] + \ldots \end{aligned}$$

plus constraints that must be satisfied by the data on the initial surface,

$$\begin{aligned} 0 &= R^{(3)} + K^2 - K_{ij} K^{ij} \\ 0 &= D_j K^{ij} - D^i K \end{aligned}$$