The initial value problem of general relativity

or

around and around and around

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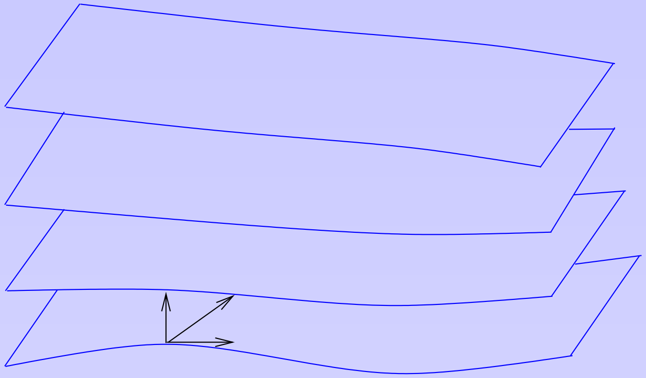
Introduction

3+1 decomposition:

- **evolution equations**

$$\partial_t \bar{g}_{ij} = -2\bar{N}\bar{K}_{ij} + \bar{\nabla}_i\beta_j + \bar{\nabla}_j\beta_i$$

$$\partial_t \bar{K}_{ij} = \ldots$$



- **constraint equations**

$$\boxed{\begin{aligned} &\bar{R} + \tau^2 - \bar{K}_{ij}\bar{K}^{ij} = 0 \\ &\bar{\nabla}_j\left(\bar{K}^{ij} - \bar{g}^{ij}\tau\right) = 0 \end{aligned}}$$

  - restrict 4 of the 12 DoF in $\{\bar{g}_{ij}, \bar{K}^{ij}\}$
  - no standard mathematical form

This talk: How to construct solutions $\{\bar{g}_{ij}, \bar{K}^{ij}\}$

Outline

1. **History/status quo**
    - 1944: Lichnerowicz
    - 1974: conformal TT
    - 1979: physical TT
    - 1999: conformal thin sandwich
    - Summary & Caveats
2. **2002: Improved decomposition**
    - step 1: $\bar{\sigma}$
    - consequences
    - step 2: $\bar{\sigma} = 2\bar{N}$
    - consequences
3. **Summary**

Basic strategy

$$\bar{R} + \tau^2 - \bar{K}_{ij}\bar{K}^{ij} = 0$$

$$\bar{\nabla}_j \left(\bar{K}^{ij} - \bar{g}^{ij}\tau \right) = 0$$

1+3 constraints on 6+6 unknowns

Split $\bar{g}_{ij}$ and $\bar{K}^{ij}$ in smaller pieces so that some pieces are completely determined, while the remaining are unconstraint

Conformal transformation of $\bar{g}_{ij}$ (Lichnerowicz, 1944)

$$\boxed{\bar{g}_{ij} = \phi^4 g_{ij}}$$

Consequences:

1. $\bar{\nabla}_j \left(\phi^{-10} S^{ij}\right) = \phi^{-10} \nabla_j S^{ij}$
2. $(\bar{\mathbb{L}}V)^{ij} = \phi^{-4} (\mathbb{L}V)^{ij}$ $\qquad (\mathbb{L}V)^{ij} = \nabla^i V^j + \nabla^j V^i - \frac{2}{3} g^{ij} \nabla_k V^k$
3. $\bar{R} = \phi^{-4} R - 8\phi^{-5} \nabla^2 \phi$

$$\Rightarrow \quad \nabla^2 \phi - \frac{1}{8} \phi R - \frac{1}{8} \phi^5 \tau^2 + \frac{1}{8} \phi^5 \bar{K}_{ij} \bar{K}^{ij} = 0$$

Split off trace of extrinsic curvature

$$\bar{K}^{ij} = \bar{A}^{ij} + \frac{1}{3}\bar{g}^{ij}\tau$$

$$\bar{R} + \frac{2}{3}\tau^2 - \bar{A}_{ij}\bar{A}^{ij} = 0$$

$$\bar{\nabla}_j\left(\bar{A}^{ij} - \frac{2}{3}\bar{g}^{ij}\tau\right) = 0$$

With $\bar{g}_{ij} = \phi^4 g_{ij}$, Hamiltonian constraint becomes

$$\nabla^2\phi - \frac{1}{8}\phi R - \frac{1}{12}\phi^5\tau^2 + \frac{1}{8}\phi^5\bar{A}_{ij}\bar{A}^{ij} = 0$$

Question: How to decompose $\bar{A}^{ij}$?

Physical TT decomposition (O'Murchadha & York, 1974)

- Decompose

$$\boxed{\bar{A}^{ij} = \bar{A}^{ij}_{TT} + (\bar{\mathbb{L}}V)^{ij}}$$

(Momentum constraint $\bar{\nabla}_j(\bar{\mathbb{L}}V)^{ij} - \frac{2}{3}\bar{g}^{ij}\partial_j\tau = 0$)

- Conformally scale to rewrite in terms of *known* g_{ij}:

$$\bar{A}^{ij} = \phi^{-10}A^{ij}_{TT} + \phi^{-4}(\mathbb{L}V)^{ij}$$

- Constraints become

$$\nabla_j(\mathbb{L}V)^{ij} + 6(\mathbb{L}V)^{ij}\partial_j \ln\phi - \frac{2}{3}g^{ij}\partial_j\tau = 0$$

$$\begin{aligned} 8\nabla^2\phi - \phi R - \frac{2}{3}\phi^5\tau^2 + \phi^{-7}A^{ij}_{TT}A^{TT}_{ij} & \\ +2\phi^{-1}A^{ij}_{TT}(\mathbb{L}V)_{ij} + \phi^5(\mathbb{L}V)^{ij}(\mathbb{L}V)_{ij} &= 0 \end{aligned}$$

These elliptic equations determine (ϕ, V^i); free data: g_{ij}, A^{ij}_{TT}, τ

Recipe to generate valid initial data

1. Choose free data $g_{ij}, A^{ij}_{TT}, \tau$
2. Solve elliptic equations for (ϕ, V^i)

$$\nabla_j(\mathbb{L}V)^{ij} + 6(\mathbb{L}V)^{ij}\partial_j \ln\phi - \frac{2}{3}g^{ij}\partial_j\tau = 0$$

$$8\nabla^2\phi - \phi R - \frac{2}{3}\phi^5\tau^2 + \phi^{-7}A^{ij}_{TT}A^{TT}_{ij}$$
$$+2\phi^{-1}A^{ij}_{TT}(\mathbb{L}V)_{ij}+\phi^5(\mathbb{L}V)^{ij}(\mathbb{L}V)_{ij} = 0$$

3. Assemble physical solution

$$\bar{g}_{ij} = \phi^4 g_{ij}$$
$$\bar{A}^{ij} = \phi^{-10}A^{ij}_{TT} + \phi^{-4}(\mathbb{L}V)^{ij}$$
$$\bar{K}^{ij} = \bar{A}^{ij} + \frac{1}{3}\phi^{-4}g^{ij}\tau$$

– Complicated, strongly coupled equations –

– Wrong sign in momentum constraint –

Conformal TT decomposition (York, 1979)

First conformally scale, *then* decompose:

$$\boxed{\bar{A}^{ij} = \phi^{-10} A^{ij}, \qquad A^{ij} = A^{ij}_{TT} + (\mathbb{L}V)^{ij}}$$

Constraints become

$$\begin{aligned} \nabla_j (\mathbb{L}V)^{ij} - \frac{2}{3}\phi^6 g^{ij}\partial_j \tau &= 0 \\ \nabla^2\phi - \phi R - \frac{1}{12}\phi^5\tau^2 + \frac{1}{8}\phi^{-7}A^{ij}A_{ij} &= 0 \end{aligned}$$

If $\tau =$ const, eqns decouple. Further, for $g_{ij} =$ flat, analytic solutions of momentum constraint known (Bowen-York extrinsic curvature). Very popular because of these simplifications.

Comments

- Physical TT $\qquad \bar{A}^{ij} = \phi^{-10} A^{ij}_{TT} + \phi^{-4} (\mathbb{L}V)^{ij}$
- Conformal TT $\qquad \bar{A}^{ij} = \phi^{-10} \left(A^{ij}_{TT} + (\mathbb{L}V)^{ij} \right)$

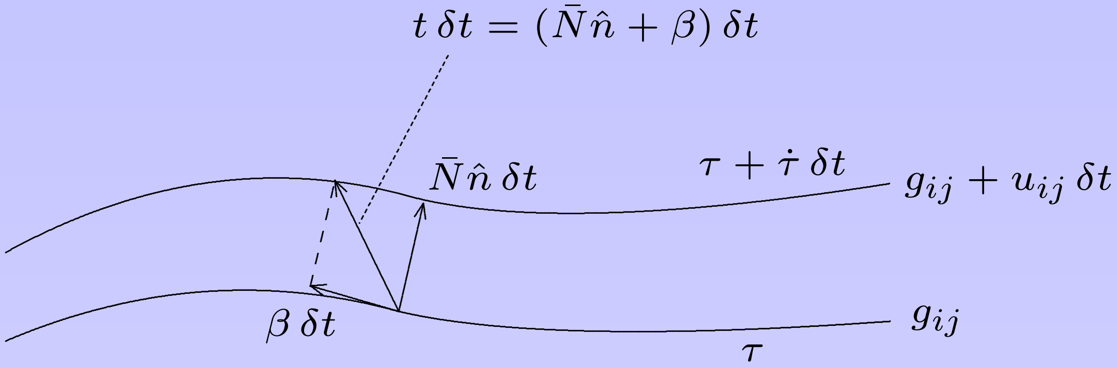
1. Both decompositions can generate any initial data set

2. Reasons for conformal scalings
   - $\bar{A}^{ij}_{TT} = \phi^{-10} A^{ij}_{TT}$: preserves divergence
   - τ unscaled (time)
   - V^i unscaled (conformal Killing operator $\mathbb{L}$)
   - $\bar{A}^{ij} = \phi^{-10} A^{ij}$ not well founded

3. Decompositions differ because TT-decomposition **does not commute** with conformal transformation
   — TT-part and longitudinal part have **different** natural scalings —

4. $\bar{K}^{ij}$ closely related to momentum conjugate to $\bar{g}_{ij}$: **Hamiltonian picture**
   Specification of $\bar{g}_{ij}$ and $\partial_t g_{ij}$ corresponds to **Lagrangian picture** (next)

Conformal thin sandwich (York, 1999)



specify:

- conformal 3-metric g_{ij}
- its velocity $\partial_t g_{ij} = u_{ij}$ (tracefree)
- mean curvature τ
- *conformal* lapse N (or $\dot{\tau}$)

lapse transforms as $\bar{N} = \phi^6 N$

- Hyperbolic systems: $\alpha = g^{-1/2}\bar{N}$ is arbitrary, not $\bar{N}$ (Choquet-Bruhat, York; KST)
- Teitelboims path integrals, Ashtekar variables simplify when one uses α instead of $\bar{N}$
- α is multiplier of Hamiltonian constraint in action principle, not $\bar{N}$ (York)
- O'Murchadha's "emergent time"

$\Rightarrow$ α conformally invariant $\Rightarrow$ $\bar{N} = \phi^6 N$

Conformal thin sandwich

$$\bar{g}_{ij} = \phi^4 g_{ij}$$

$\partial_t \quad \Rightarrow$ $$-2\bar{N}\bar{K}_{ij} + \bar{\nabla}_i\beta_j + \bar{\nabla}_j\beta_i = 4\phi^3(\partial_t\phi)g_{ij} + \phi^4 u_{ij}$$

trace free piece $\Rightarrow$ $$-2\bar{N}\bar{A}_{ij} + (\bar{\mathbb{L}}\beta)_{ij} = \phi^4 u_{ij}$$

rearrange $\Rightarrow$ $$\bar{A}^{ij} = \frac{1}{2\bar{N}}\Big((\bar{\mathbb{L}}\beta)^{ij} - \underbrace{\phi^{-4}\,u^{ij}}_{\bar{u}^{ij}}\Big)$$

rewrite in conformal quantities:

$$\bar{A}^{ij} = \frac{\phi^{-4}}{2\phi^6 N}\Big((\mathbb{L}\beta)^{ij} - u^{ij}\Big) \quad = \frac{\phi^{-10}}{2N}\Big((\mathbb{L}\beta)^{ij} - u^{ij}\Big) = \phi^{-10}A^{ij}$$

- Decomposition of $\bar{A}^{ij}$ is not assumed, but follows from specification of g_{ij} and u^{ij}.
- A^{ij} form-invariant under conformal transformation
- Scaling $\bar{A}^{ij} = \phi^{-10}A^{ij}$ *consequence*, not *assumption*

Conformal thin sandwich cont'd

$$\bar{A}^{ij} = \frac{1}{2\bar{N}}\left((\bar{\mathbb{L}}\beta)^{ij} - \bar{u}^{ij}\right) = \frac{\phi^{-10}}{2N}\left((\mathbb{L}\beta)^{ij} - u^{ij}\right) = \phi^{-10}A^{ij}$$

- substitute into constraints ⇒ elliptic eqns for ϕ, β^i

$$\nabla_j\left(\frac{1}{2N}(\mathbb{L}\beta)^{ij}\right) - \nabla_j\left(\frac{1}{2N}u^{ij}\right) - \frac{2}{3}\phi^6\tilde{\nabla}^i\tau = 0$$

$$8\nabla^2\phi - \phi R - \frac{2}{3}\phi^5\tau^2 + \phi^{-7}A^{ij}A_{ij} = 0$$

- u^{ij} has five DoF, whereas previously, A^{ij}_{TT} has only two.
  The three additional DoF determine a particular choice of shift (gauge):

$$u^{ij} \rightarrow u^{ij} + (\mathbb{L}W)^{ij} \;\Rightarrow\; \beta^i \rightarrow \beta^i + W^i$$

- If one specifies $\dot{\tau}$ instead of $\tilde{N}$, then free data is symmetric: $(g_{ij}, \dot{g}_{ij},\ \tau, \dot{\tau})$

Conformal transformations of free data

– Conformal superspace should allow conformal transformations of free data –

Example: **Conformal thin sandwich**

1. Given g_{ij}, u^{ij}, N, τ, solve constraints, find physical solution

$$\bar{g}_{ij} = \phi^4 g_{ij}$$

$$\bar{A}^{ij} = \phi^{-10} \frac{1}{2N} \left(u^{ij} + (\mathbb{L}\beta)^{ij} \right)$$

2. Define *scaled* free data $\hat{g}_{ij} = \rho^{-4} g_{ij}, \hat{u}^{ij} = \rho^4 u^{ij}, \hat{N} = \rho^{-6} N$.
3. $(\phi\rho, \beta^i)$ in hatted free data leads to *same* physical initial data:

$$\bar{g}_{ij} = \phi^4 \rho^4 \hat{g}_{ij} = \phi^4 g_{ij}$$

$$\bar{A}^{ij}_{\text{hat}} = (\phi\rho)^{-10} \frac{1}{2\hat{N}} \left(\hat{u}^{ij} + (\hat{\mathbb{L}}\beta)^{ij} \right) = (\phi\rho)^{-10} \frac{1}{2\rho^{-6} N} \left(\rho^4 u^{ij} + \rho^4 (\mathbb{L}\beta)^{ij} \right) = \bar{A}^{ij}_{\text{tilde}}$$

Solution of conformal thin sandwich invariant

Conformal transformations on free data cont'd

Conformal TT not invariant:

$$\bar{A}^{ij} = \phi^{-10} A^{ij}_{TT} + \phi^{-10} (\mathbb{L}V)^{ij}$$

Scaled free data $\hat{g}_{ij} = \rho^{-4} g_{ij}$, $\hat{A}^{ij}_{TT} = \rho^{10} A^{ij}_{TT}$.

Require $\bar{A}^{ij} = \bar{A}^{ij}$:

$$\phi^{-10} A^{ij}_{TT} + \phi^{-10} (\mathbb{L}V)^{ij} = (\phi\rho)^{-10} \hat{A}^{ij}_{TT} + (\phi\rho)^{-10} \rho^4 (\mathbb{L}\hat{V})^{ij}$$

$$(\mathbb{L}\hat{V})^{ij} = \rho^6 (\mathbb{L}V)^{ij}$$

Three unknowns $\hat{V}^i$ must satisfy five equations (unlikely to work). Definitely, $\hat{V}^i = V^i$ will not work.

Physical TT invariant under $g_{ij} = \rho^{-4} g_{ij}$, $\hat{A}^{ij}_{TT} = \rho^{10} A^{ij}_{TT}$.

— Same invariance properties in the **"guess-and-correct"** picture used by Matzner et al, Tichy et al, Pfeiffer et al. —

Summary of status quo

- Physical TT $\bar{A}^{ij} = \phi^{-10} A^{ij}_{TT} + \phi^{-4} (\mathbb{L}V)^{ij}$
- Conformal TT $\bar{A}^{ij} = \phi^{-10} \left(A^{ij}_{TT} + (\mathbb{L}V)^{ij} \right)$
- Conf. thin sandwich $\bar{A}^{ij} = \phi^{-10} \frac{1}{2N} \left(u^{ij} + (\mathbb{L}\beta)^{ij} \right)$

1. two inequivalent Hamiltonian pictures
2. both inequivalent to (Lagrangian) conformal thin sandwich
3. Lagrangian formulation invariant under conf. transformations of free data
4. Conformal TT not invariant
5. Physical TT invariant, but leads to less appealing PDEs

$\Rightarrow$ **Hamiltonian picture not yet complete**

Improved Hamiltonian picture (step 1)

Introduce weight function $\bar{\sigma} > 0$ and decompose

$$\bar{A}^{ij} = \bar{A}^{ij}_{TT} + \frac{1}{\bar{\sigma}}(\bar{\mathbb{L}}V)^{ij}.$$

Require conformal scaling $\boldsymbol{\bar{\sigma} = \phi^6 \sigma}$. Then

$$\bar{A}^{ij} = \bar{A}^{ij}_{TT} + \bar{\sigma}^{-1}(\bar{\mathbb{L}}V)^{ij} = \phi^{-10}\left(A^{ij}_{TT} + \sigma^{-1}(\mathbb{L}V)^{ij}\right) = \phi^{-10}A^{ij}.$$

— Decomposition of A^{ij} *form-invariant* under conformal transformation
— $\bar{A}^{ij} = \phi^{-10}A^{ij}$ *follows* from scaling of σ

Momentum constraint becomes

$$\nabla_j\left(\sigma^{-1}(\mathbb{L}V)^{ij}\right) - \frac{2}{3}\phi^6\nabla^i\tau = 0$$

— Elliptic equation if σ is given

Improved Hamiltonian picture (after step 1)

- Free data $g_{ij}, A^{ij}_{TT}, \tau$ *and* σ
- Constraints

$$\nabla_j\left(\sigma^{-1}(\mathbb{L}V)^{ij}\right) - \frac{2}{3}\phi^6\nabla^i\tau = 0,$$

$$\nabla^2\phi - \frac{1}{8}\phi R - \frac{1}{12}\phi^5\tau^2 + \frac{1}{8}\phi^{-7}A^{ij}A_{ij} = 0$$

with $A^{ij} = A^{ij}_{TT} + \sigma^{-1}(\mathbb{L}V)^{ij}$

Some consequences:

1. $\nabla_j\big(\sigma^{-1}(\mathbb{L}V)^{ij}\big)$ well-behaved elliptic operator
2. "Good" signs for maximum principle in Hamiltonian constraint
3. Fairly simple eqns, decouple for τ=const
4. σ *parametrizes* TT-decomposition
   - $\sigma \equiv 1 \Rightarrow$ Conformal TT
   - $\bar\sigma \equiv 1 \Rightarrow$ Physical TT (but $\bar\sigma$ not freely specificable)

More consequences: Conformal invariance

free data $g_{ij},\ A^{ij}_{TT},\ \tau,\ \sigma$ gives solution (ϕ, V^i). Get **same** physical solution with scaled free data

$$\hat{g}_{ij} = \rho^{-4} g_{ij}, \quad \hat{A}^{ij}_{TT} = \rho^{10} A^{ij}_{TT}, \quad \tau, \quad \hat{\sigma} = \rho^{-6}\sigma.$$

Proof: $(\phi\rho, V^i)$ in the hatted free data yields the same physical $\bar{g}_{ij}$ and $\bar{A}^{ij}$:

$$\bar{g}_{ij} = (\phi\rho)^4 \hat{g}_{ij} = \phi^4 g_{ij}$$

$$\bar{A}^{ij} = (\phi\rho)^{-10}\left(\hat{A}^{ij}_{TT} + \hat{\sigma}^{-1}(\hat{\mathbb{L}}V)^{ij}\right) = \phi^{-10}\left(A^{ij}_{TT} + (\mathbb{L}V)^{ij}\right)$$

Conformal invariance holds also in the "guess-and-correct" picture (because σ synchronizes conformal transformations of TT and longitudinal pieces)

Improved Hamiltonian picture (step 2)

Identify $\bar{\sigma}$ with lapse $\bar{N}$

$$\bar{\sigma} = 2\bar{N}, \qquad\qquad \sigma = 2N$$

Reasons:

1. They have same conformal scaling.
2. Hamiltonian picture agrees with Lagrangian picture (next slide)
3. Stationary spacetimes have vanishig TT-piece of $\bar{K}^{ij}$ (2nd to next slide)

Equivalence of weighted decomposition and thin sandwich

In general, construct A_{TT}^{ij} by decomposing trace-free symmetric tensor M^{ij}

$$\begin{aligned} M^{ij} &= A_{TT}^{ij} + \sigma^{-1}(\mathbb{L}W)^{ij}, \\ \nabla_j M^{ij} &= \nabla_j \left(\sigma^{-1}(\mathbb{L}W)^{ij}\right), \\ A_{TT}^{ij} &= M^{ij} - \sigma^{-1}(\mathbb{L}W)^{ij}. \end{aligned}$$

Decomposition of $\bar{A}^{ij}$ now becomes ($\sigma = 2N$)

$$\bar{A}^{ij} = \phi^{-10}\left(M^{ij} + (2N)^{-1}\left(\mathbb{L}(V-W)\right)^{ij}\right)$$

Compare to conformal thin sandwich

$$\bar{A}^{ij} = \frac{\phi^{-10}}{2N}\left((\mathbb{L}\beta)^{ij} - u^{ij}\right)$$

Identification $M^{ij} = -(2N)^{-1}u^{ij}$ and $V^i - W^i = \beta^i$ results in *identical* formulae for $\bar{A}^{ij}$ $\Rightarrow$ **decompositions equivalent.**

Stationary spacetimes have $\bar{A}^{ij}_{TT} = 0$

Given: Slice Σ through a stationary spacetime with Killing vector t^μ.

Pick time-vector in evolution along t: $\bar{N} = -t \cdot n$, $\beta =\perp t$. Then

$$0 = \partial_t \bar{g}_{ij} = -2\bar{N}\bar{K}_{ij} + \bar{\nabla}_i\beta_j + \bar{\nabla}_j\beta_i$$

Removing the trace yields

$$\bar{A}^{ij} = \frac{1}{2\bar{N}}(\bar{\mathbb{L}}\beta)^{ij}$$

New decomposition has zero TT-part: $\bar{A}^{ij}_{TT} = 0$ independent of slice.

Without weight, $\bar{A}^{ij} = \bar{Q}^{ij}_{TT} + (\bar{\mathbb{L}}V)^{ij}$ has in general $\bar{Q}^{ij}_{TT} \neq 0$.

Zero $\bar{A}^{ij}_{TT}$ for stationary spacetimes meshes nicely with identification of $\bar{A}^{ij}_{TT}$ as *radiative degrees*.

Orthogonality

“Old” decomposition $\bar{M}^{ij} = \bar{M}^{ij}_{TT} + (\bar{\mathbb{L}}V)^{ij}$ orthogonal in measure $\sqrt{\bar{g}}$

$$\int \bar{M}^{ij}_{TT}(\bar{\mathbb{L}}V)^{ij}\ \sqrt{\bar{g}}\, d^3x = 0$$

“New” decomposition $\bar{M}^{ij} = \bar{M}^{ij}_{TT} + (2\bar{N})^{-1}(\bar{\mathbb{L}}V)^{ij}$ orthogonal in measure $\bar{N}\sqrt{\bar{g}} = \sqrt{{}^{(4)}\bar{g}}$:

$$\int \bar{M}^{ij}_{TT}\frac{1}{2\bar{N}}(\bar{\mathbb{L}}V)^{ij}\ \bar{N}\sqrt{\bar{g}}\, d^3x = 0$$

Summary

In the “Hamiltonian picture” of the initial value problem, one should decompose the extrinsic curvature as

$$\bar{K}^{ij} = \bar{A}^{ij}_{TT} + \frac{1}{2\bar{N}}(\bar{\mathbb{L}}V)^{ij} + \frac{1}{3}\bar{g}^{ij}\tau$$

where $\bar{N}$ scales as $\bar{N} = \phi^6 N$. This results in

1. Equivalence of Hamiltonian and Lagrangian pictures (extrinsic curvature formulation & conformal thin sandwich)
2. Invariance of physical initial data under conformal rescalings of free data
3. $\bar{A}^{ij}_{TT} = 0$ for stationary spacetimes