# Binary black hole initial data based on post-Newtonian data

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## Plan of the talk:

- Overview about the Post-Newtonian (PN) theory used
- Introducing black holes (BH) into the data
- Solving the constraint equations of General Relativity (GR) with the help of the York procedure
- Numerical results (ADM masses)
- Ambiguity in applying the York procedure
- How to use this ambiguity to minimize the changes introduced by solving the constraint equations of GR
- Discussion

# Post-Newtonian (PN) theory:

PN-theory is an expansion in a small parameter

$$\epsilon \sim v/c \sim \sqrt{M/r}$$

## Gauge choice:

We use expressions for 3-metric and extrinsic curvature computed by Jaranowski and Schäfer in the ADMTT gauge, which is specified by

$$g_{ij}^{PN} = \psi_{PN}^4 \delta_{ij} + h_{ij}^{TT}$$

$$\pi_{PN}^{ij} \delta_{ij} = 0$$

The ADMTT gauge has several advantages:

- we can easily find expressions for 3-metric and extrinsic curvature
- unlike in the harmonic gauge no logarithmic divergences appear
- for a single black hole (BH) the data simply reduce to Schwarzschild in standard isotropic coordinates
- up to $\epsilon^3$ the data look like in the puncture approach, which simplifies calculations
- for large separations we recover Brill-Lindquist data
- the trace of the extrinsic curvature vanishes up to order $\epsilon^6$ → can decouple Hamiltonian and momentum constraint equations

## **PN theory** (continued):

Limitations:

- PN theory deals with point particles, NOT black holes (BH)

→ One has to somehow introduce BHs into the theory

- The PN perturbation expansion is only valid if $\epsilon \sim v/c \sim \sqrt{M/r} \ll 1$

→ PN breaks down close to each particle or when particles get close to each other

- at the moment the particles have no spin

- Jaranowski and Schäfer's expressions are near zone expansions

→ expressions become inaccurate in the wave zone (need $r \ll \lambda \sim \pi\sqrt{r_{12}^3/(m_1+m_2)}$)

→ We have PN data which are valid only in a limited region of space.

⋆ So far the computational grid does not extend into the wave zone → near zone approximation is valid!

## **PN theory** (continued):

We use data computed by Jaranowski and Schäfer up to $O(\epsilon^5)$ for 3-metric and extrinsic curvature:

$$g_{ij}^{PN} = \psi_{PN}^4 \delta_{ij} + \epsilon^4 h^{TT}_{ij(4)} + \epsilon^5 h^{TT}_{ij(5)} + O(\epsilon^6)$$

$$K^{ij}_{PN} = \psi_{PN}^{-10} \left[ \epsilon^3 K^{ij}_{BY} - \epsilon^5 \frac{1}{2} \dot{h}^{TT}_{ij(4)} - \epsilon^5 (\phi_{(2)} \tilde{\pi}^{ij}_{(3)})^{TT} \right] + O(\epsilon^6)$$

- $\psi_{PN} = 1 + \epsilon^2 \sum_{A=1}^{2} \frac{E_A}{2r_A} + O(\epsilon^6)$, where $E_A = m_A + \epsilon^2 \left( \frac{p_A^2}{2m_A} - \frac{m_1 m_2}{2r_{12}} \right)$

- $\psi_{PN}$ and $h^{TT}_{ij(4)} + h^{TT}_{ij(5)}$ diverge like $1/r_A$ near BH

→ $g_{ij}^{PN}$ diverges at the BHs

- $h^{TT}_{ij(4)}$ and $h^{TT}_{ij(5)}$ are near zone expansions, all other terms are globally valid

- $K^{ij}_{BY} \sim p/r^2$ is of Bowen-York form

- $K^{ij}_{PN}$ is finite at BH location, because of factor $\psi_{PN}^{-10}$

- $K_{PN} \sim O(\epsilon^7) \rightarrow$ maximal slicing up to $O(\epsilon^6)$, which can be used to decouple the Hamiltonian constraint equation from the momentum constraint equations.

- PN data are similar to puncture data

# BHs in the PN data?

- near each particle the 3-metric is approximated by

$$g_{ij}^{PN} \approx \left(1 + \frac{E_A}{2r_A}\right)^4 \delta_{ij} + O(1/r_A^3)$$

- for $r_A \to 0$ this is the Schwarzschild metric in isotropic coordinates (puncture representation)

→ we do have a black hole centered on each particle

- **Reason:**
  we have not fully expand $\psi_{PN}^4 = \left(1 + \epsilon^2 \sum_{A=1}^{2} \frac{E_A}{2r_A}\right)^4$
  - if we also expand $\psi_{PN}^4$ in $g_{ij}^{PN}$

$$g_{ij}^{PN} \approx \left(\frac{\text{const}}{r_A^2}\right) \delta_{ij} + O(1/r_A)$$

  → puncture singularity of Schwarzschild is gone

- from now on we will use the first (unexpanded) form of the 3-metric in order to introduce BHs into the PN data

- i.e. the point particle singularity is replaced by the puncture coordinate singularity

- this choice is somewhat ad hoc
- YET, PN is not valid near the particles anyway
- the choice of putting in BHs as punctures seems natural

## Constraints

BH initial data in GR have to fulfill the Hamiltonian and momentum constraint equations:

$$R + K^2 - K_{ij}K^{ij} = 0$$

$$\nabla_j(K^{ij} - Kg^{ij}) = 0$$

- the pure PN data as given by Jaranowski and Schäfer fulfill the constraints of GR only up to the PN order used

→ pure PN data violate the constraints, especially close to the BHs where PN theory breaks down

- we have to modify the data such that they fulfill the constraints
- we want to keep the modifications small in the region where PN theory is a good approximation, i.e. far away from the BHs.
- we will use the York procedure to project the pure PN data onto the solution manifold of GR
- we will show later how large the differences before and after applying the York procedure are, and how to minimize them

# York procedure

In order to find a $g_{ij}$ and $K^{ij}$ which fulfill the Constraints of GR we use the York procedure:

- solve the elliptic equations

$$\Delta^{g^{PN}}\Psi - \frac{1}{8}\Psi R_{g^{PN}} + \ldots = 0 \quad \text{and} \quad \Delta_L^{g^{PN}} W^i + \nabla_j K^{ij}_{PN} = 0$$

  for $\Psi$ and $W^i$

$\rightarrow$ $g_{ij} = \Psi^4 g^{PN}_{ij}$ and $K^{ij} = \Psi^{-10}[(LW)^i + K^{ij}_{PN}]$

But Problem: $\psi_{PN} \sim O(1/r)$ near BH

$$\rightarrow \quad g^{PN}_{ij} \sim O(1/r^4) \quad , \quad \Gamma^i_{jk} \sim O(1/r^5) \quad , \quad R_{g^{PN}} \sim O(1/r^6)$$

$\rightarrow$ Divergences in elliptic eqs, which cause trouble numerically

Solution: rescale $g^{PN}_{ij} \rightarrow \psi^{-4}_{PN} g^{PN}_{ij}$ and $K^{ij}_{PN} \rightarrow \psi^{10}_{PN} K^{ij}_{PN}$

- $g^{PN}_{ij}$, $\Gamma^i_{jk}$ and $R_{g^{PN}}$ are now regular
- but expect $\Psi \approx \psi_{PN}$ $\rightarrow$ new divergences!
- use $\Delta^\delta \psi_{PN} = 0$ and make Ansatz: $\Psi = \psi_{PN} + u$

  $\rightarrow$ $\Delta^{g^{PN}}\Psi = \Delta^\delta u + \ldots$

  $\rightarrow$ u is finite and does not cause trouble
- similarly we can use $\partial_j K^{ij}_{BY} = 0$ to simplify $\nabla_j K^{ij}_{PN}$

$\rightarrow$ We find elliptic eqs which can be solved numerically!

## Numerics

- we use second order finite differencing together with the multigrid elliptic solver BAM_Elliptic in Cactus

- all grids have uniform resolution

- the two black hole punctures are always staggered between grid points

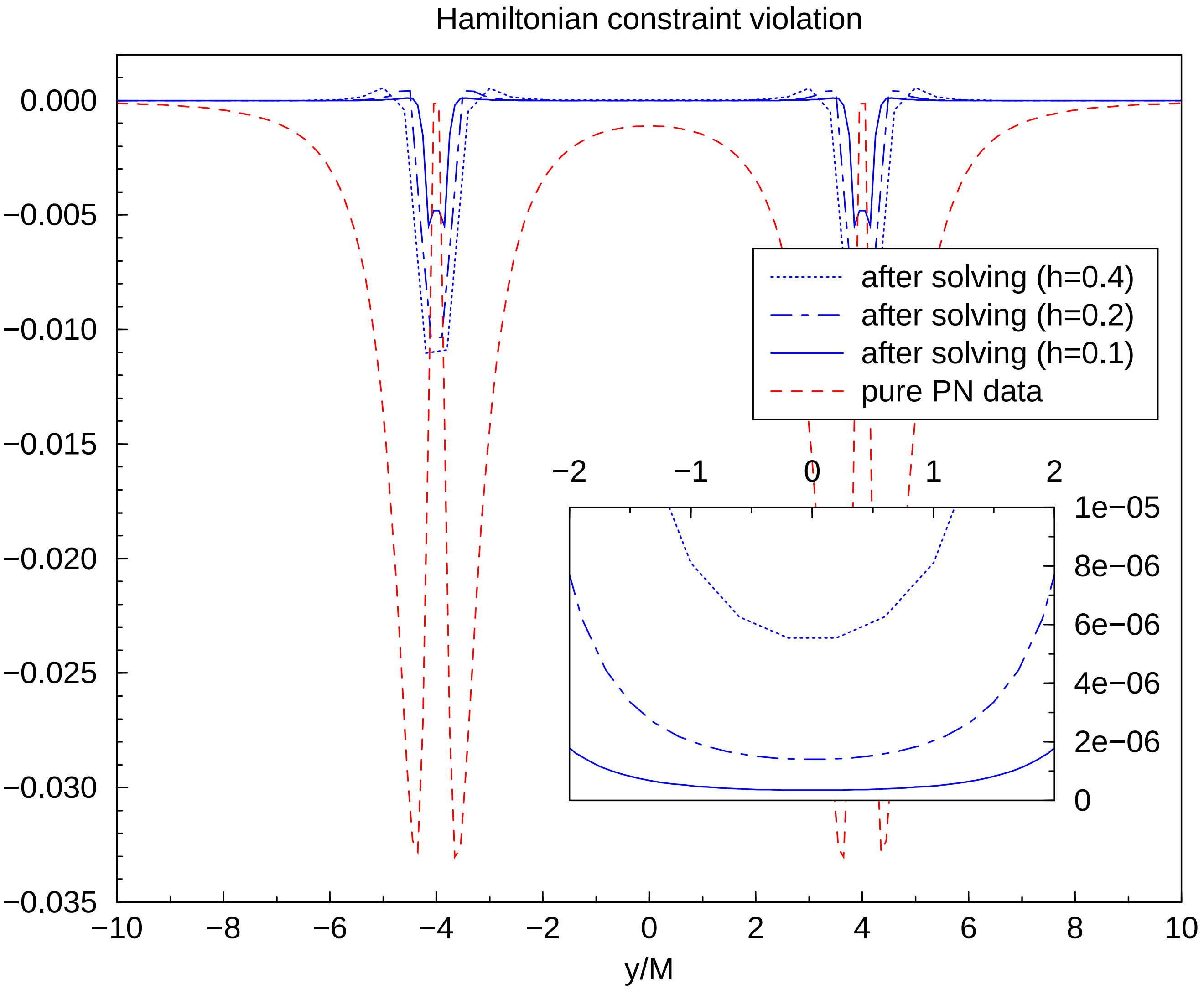
- as outer boundary conditions we use "Robin" conditions, i.e. we assume that $u \propto 1/r$ and $W^i \propto 1/r$

- we consider non-spinning equal mass binaries with their center of mass at rest at the origin

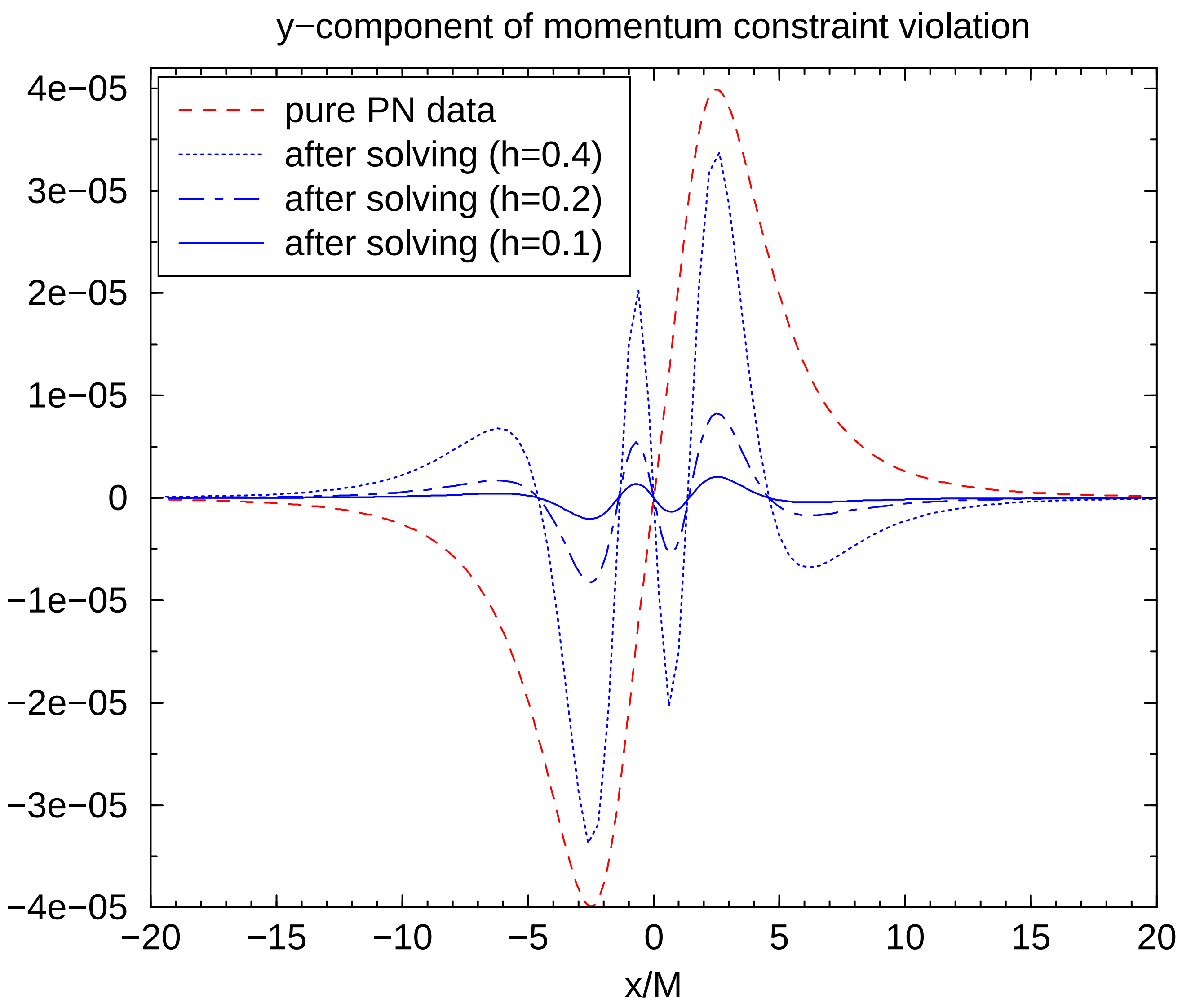
- we set the BH momenta before solving to the value for 2PN circular orbits

→ the binaries are in quasi-circular orbits

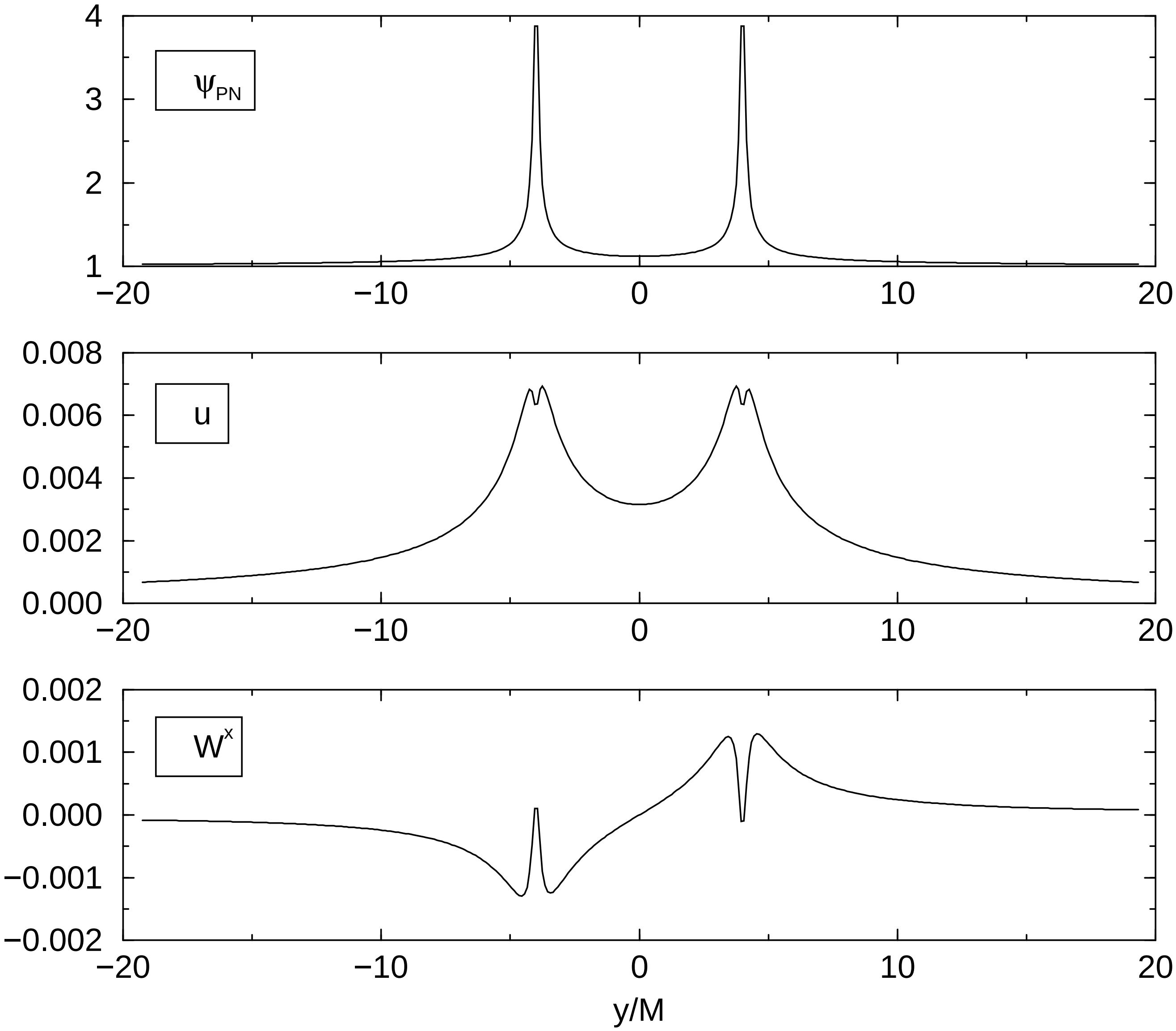
- the two BHs are on the y-axis, such that their momenta point in the positive and negative x-directions, resulting in an angular momentum along the z-direction



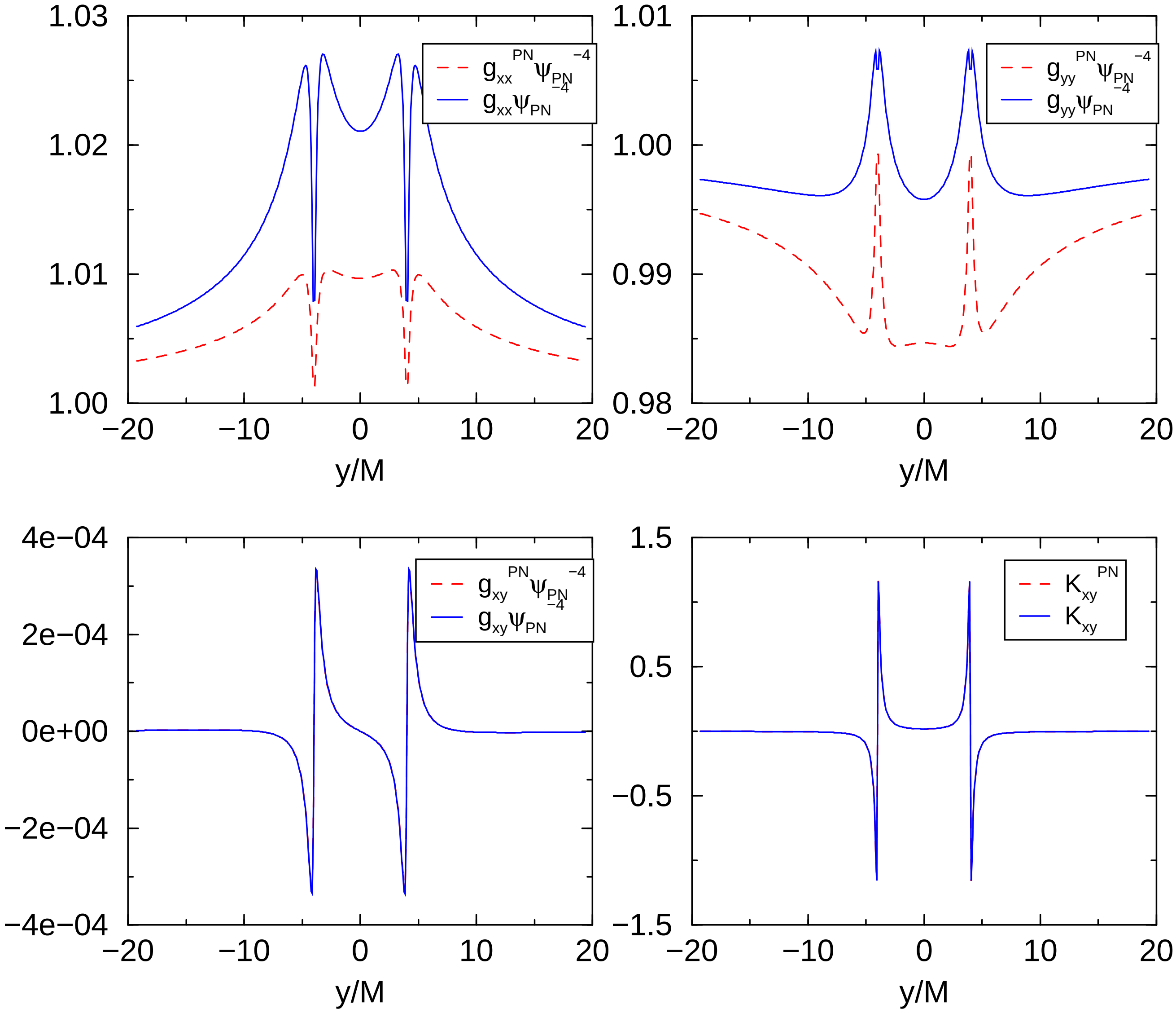
Hamiltonian constraint violation for a black hole separation of $r_{12} = 8M$. The Hamiltonian constraint of pure PN data is much larger than the Hamiltonian constraint after solving. The inset is a blow up of the central region, which shows that our numerical scheme is second order convergent as expected.



The momentum constraint for a separation of $r_{12} = 8M$. We observe second order convergence in the resolution $h$ after solving. The momentum constraint violation of pure PN data is larger than after solving.



The solutions of $u$ and $W^x$ along the y-axis for a black hole separation of $r_{12} = 8M$. For comparison we also show $\psi_{PN}$, which diverges at $y = \pm 4$.



Components of the 3-metric and extrinsic curvature for a black hole separation of $r_{12} = 8M$. The data are shown before (dashed lines) and after applying the York procedure (solid lines). The components of the 3-metric change on the order of $\sim 1\%$.

# Changes in the data due to applying the York procedure

After applying the York procedure, $g_{ij}$ and $K^{ij}$ are different from the pure PN expressions $g_{ij}^{PN}$ and $K^{ij}_{PN}$.

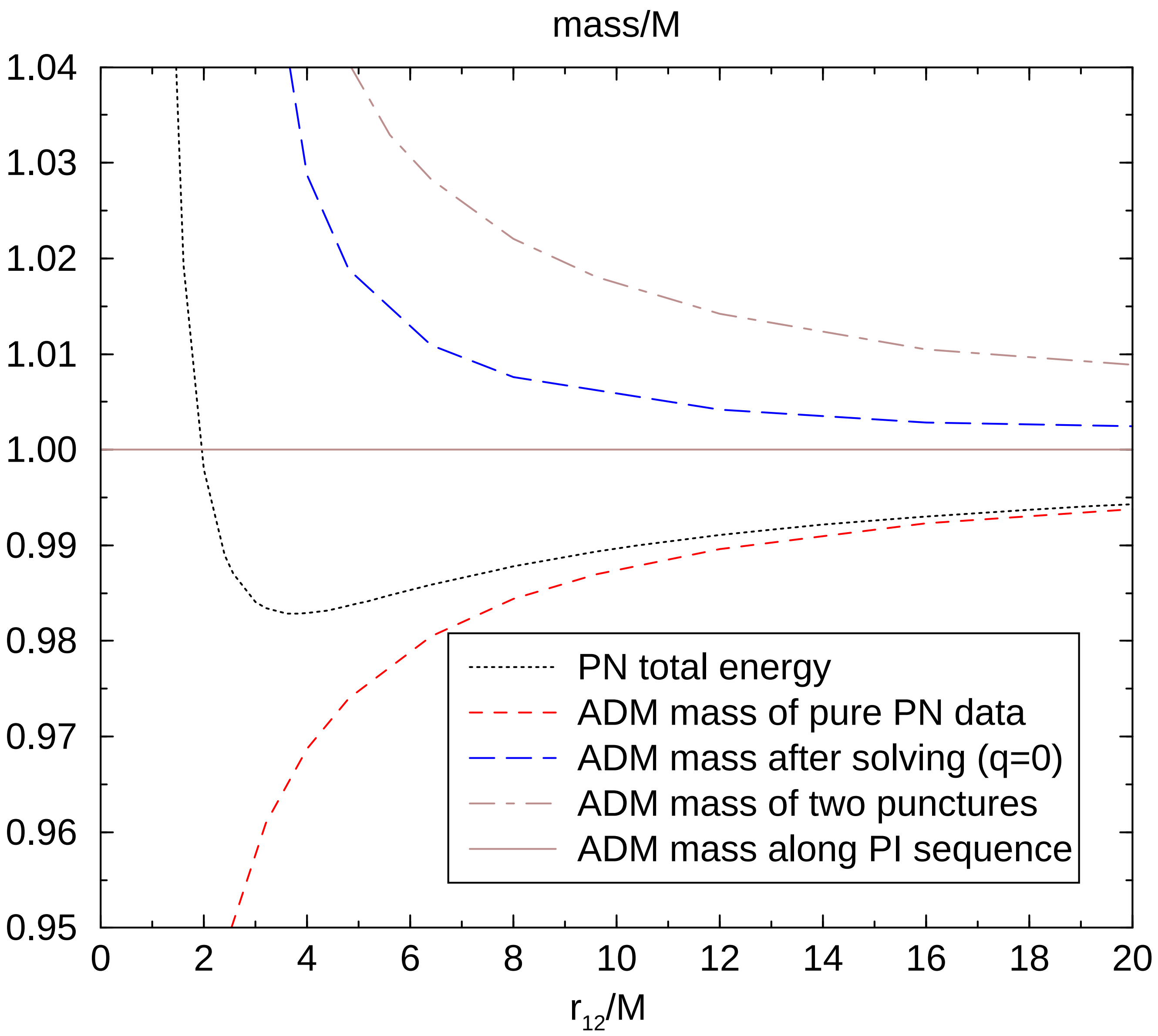
The 3-metric, extrinsic curvature and $h^{TT}_{ij(4)}$ at $x = 0$, $y = 12.2M$, $z = 0$ for two BHs located at $y = \pm 5.2M$:

| PN value (up to $O(\epsilon^5)$) | Value after solving ($q = 0$) | relative difference |
|---|---|---|
| $g^{PN}_{xx} = 1.21866$ | $g_{xx} = 1.22285$ | $\frac{g_{xx}-g^{PN}_{xx}}{g^{PN}_{xx}} = 0.0034$ |
| $K^{PN}_{xy} = -0.0022341$ | $K_{xy} = -0.0022617$ | $\frac{K_{xy}-K^{PN}_{xy}}{K^{PN}_{xy}} = -0.012$ |
| PN metric (up to $O(\epsilon^5)$) | TT term in metric of $O(\epsilon^4)$ | relative size of $O(\epsilon^4)$ correction |
| $g^{PN}_{xx} = 1.21866$ | $h^{TT}_{xx(4)} = 0.00443$ | $\frac{h^{TT}_{xx(4)}}{g^{PN}_{xx}} = 0.0036$ |

- the increase in the 3-metric due to applying the York procedure has about the same order of magnitude as the PN corrections at $O(\epsilon^4)$

- this happens in a region far enough from the particles that PN theory can actually be trusted to give realistic values

→ solving the constraint equations introduces significant differences between $g_{ij}$ and $g^{PN}_{ij}$ in the outer region due to changes in the inner region



PN inspiral sequence:

- keep particle masses $m_1$ and $m_2$ constant along sequence
- use momentum for 2PN circular orbits in the pure PN data before solving for each separation $r_{12}$

## The data change after applying the York procedure

**Main reason for change:**

- change in conformal factor $u$ is always positive

→ the new conformal factor $\Psi = \psi_{PN} + u$ computed in the York procedure is always larger than $\psi_{PN}$

→ the 3-metric increases

→ the ADM mass also increases

## Idea for compensating for changes in the data

- decrease $\psi_{PN}$ before applying the York procedure:
  - find a way to achieve $\Psi = \psi_{PN} + Q + u$
  - choose $Q$ such that $Q + u \approx 0$
    at least in the region far from the BHs where PN theory is valid

→ this way the data gets modified twice, so that in the end the two modifications almost cancel

**Questions:**

(i) How do we find a suitable $Q$?

- recall that $\psi_{PN} = 1 + \epsilon^2 \sum_{A=1}^{2} \frac{E_A}{2r_A} + O(\epsilon^6)$
- simply change $E_A$ in $\psi_{PN}$

(ii) Is it allowed to add terms to $\psi_{PN}$?

- can exploit an ambiguity in applying the York procedure to generate additional terms in $\psi_{PN}$

## Ambiguity in the York procedure

- the York procedure explained before was applied to the conformally rescaled quantities $\psi_{PN}^{-4} g_{ij}^{PN}$ and $\psi_{PN}^{10} K_{PN}^{ij}$

- there is a priori no reason for using the rescaled quantities

- in principle we could have also started directly with $g_{ij}^{PN}$ and $K_{PN}^{ij}$ or with $g_{ij}^{PN}$ and $K_{PN}^{ij}$ scaled by any function $\Omega$, i.e. with

$$\tilde{g}_{ij}^{PN} = \Omega^4 g_{ij}^{PN}$$

$$\tilde{K}_{PN}^{ij} = \Omega^{-10} K_{PN}^{ij}$$

- the York procedure then would still yield a solution to the constraints

- each of these different starting points will in general yield different results for $g_{ij}$ and $K_{ij}$

**IDEA:** Use this freedom to modify the pure PN data such that the PN conformal factor $\psi_{PN}$ decreases, in order to compensate for the increase in the overall conformal factor $\Psi$ due to applying the York procedure.

## Choosing a scale factor $\Omega$

- use $\Omega = 1 + \epsilon^4 Q + O(\epsilon^6)$

→ the PN conformal factor changes to

$$\tilde{\psi}_{PN} = \psi_{PN} + \epsilon^4 Q$$

while all other PN terms up to $O(\epsilon^5)$ remain unchanged

→ conformal factor due to York procedure is now

$$\Psi = \tilde{\psi}_{PN} + u = \psi_{PN} + \epsilon^4 Q + u$$

- we expect that $u \approx \frac{N}{r_{12} r}$ for large $r$ and $r_{12}$

→ choose

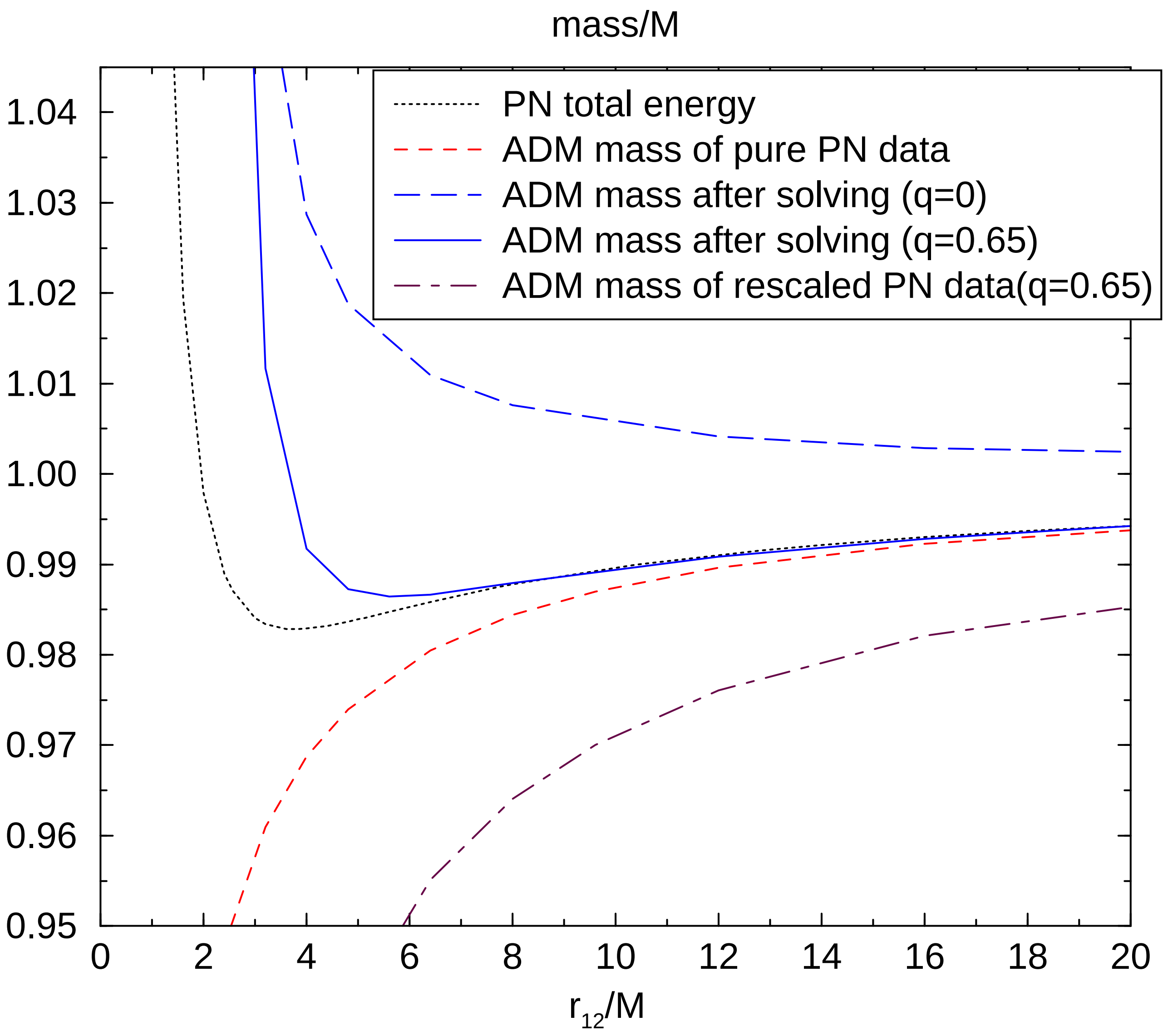
$$Q = -q \frac{m_1 m_2}{2 r_{12}} \left( \frac{1}{2 r_1} + \frac{1}{2 r_2} \right)$$

- this corresponds to a change of $E_A = m_A + \epsilon^2 \left( \frac{p_A^2}{2 m_A} - \frac{m_1 m_2}{2 r_{12}} \right)$ in $\psi_{PN} = 1 + \epsilon^2 \sum_{A=1}^{2} \frac{E_A}{2 r_A} + O(\epsilon^6)$ of the form

$$E_A \rightarrow \tilde{E}_A = E_A - \epsilon^2 q \frac{m_1 m_2}{2 r_{12}}$$

→ With this choice of $Q$ the increase in the conformal factor due to $u$ can be compensated if we choose an appropriate value for $q$.

Numerically we find that $q = 0.65$ does the job.

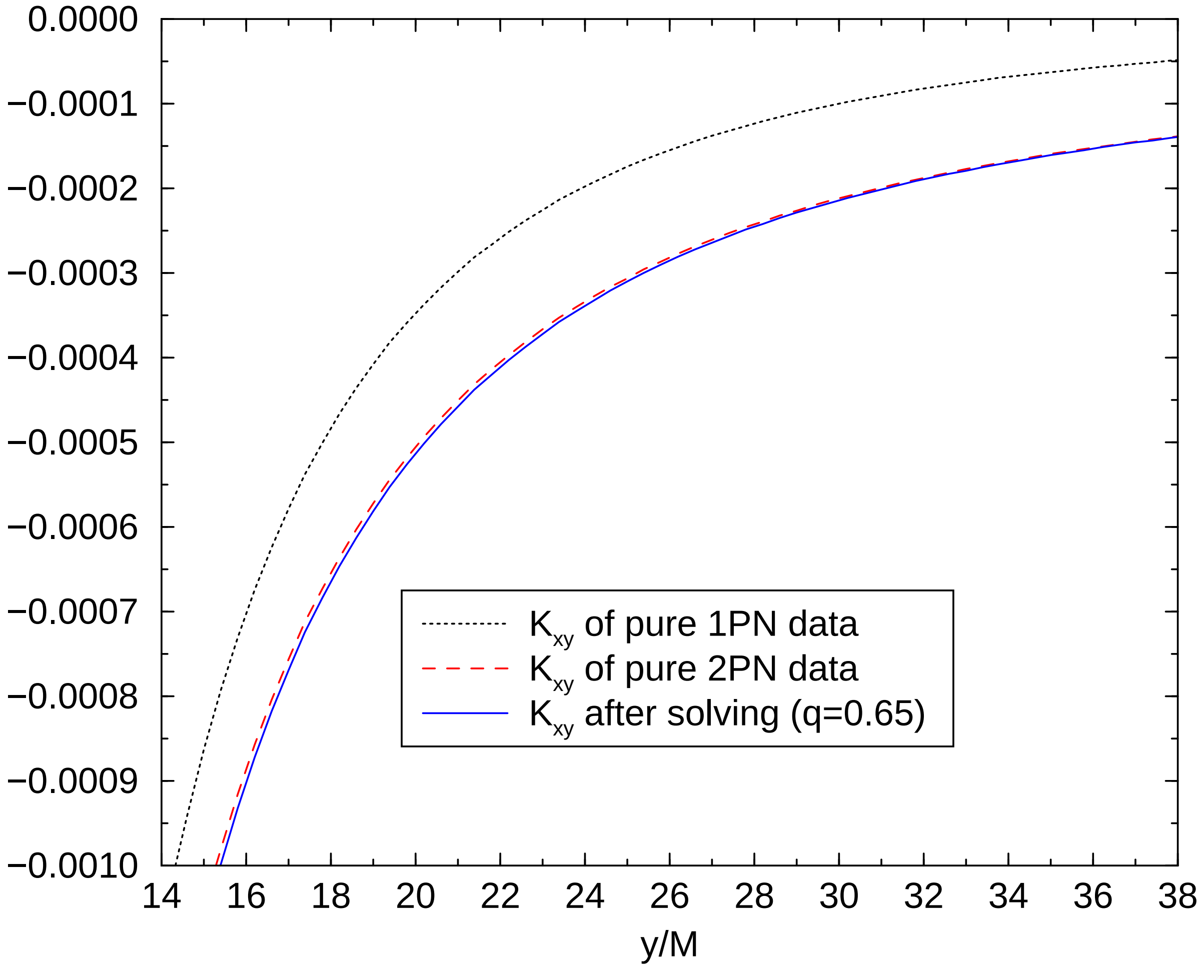


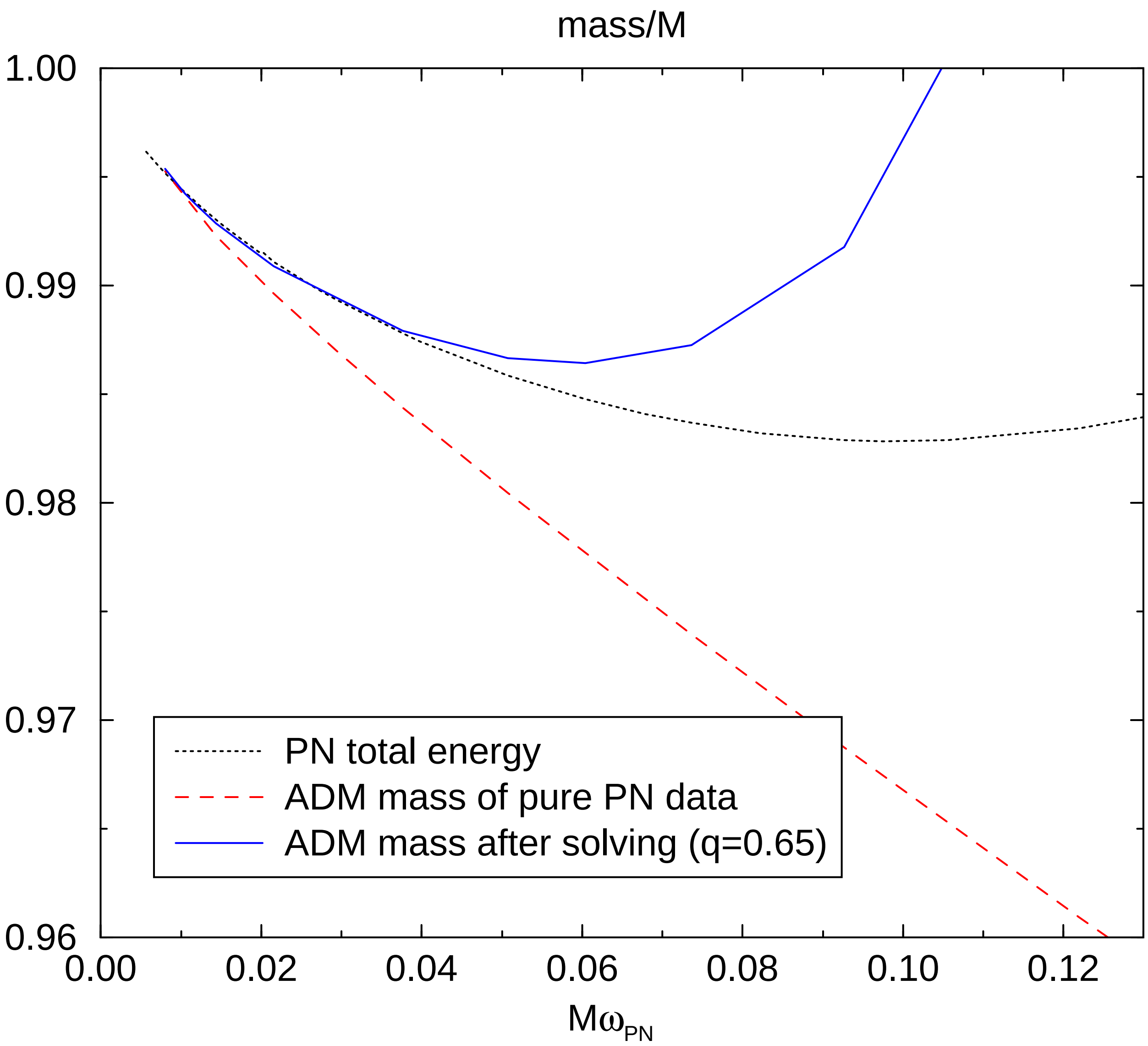
- solid blue line: ADM mass after solving with $q = 0.65$
  - + does not change very much
  - + closely follows the PN energy down to $r_{12} \approx 6M$
  - + is physically reasonable until $r_{12} \approx 5.6M$ since it decreases with decreasing separation

## Changes in the data due to applying the extended York procedure with q=0.65

In the region where PN theory is valid ($r_A \gg M$ and $r_{12} \gg M$) we find that:

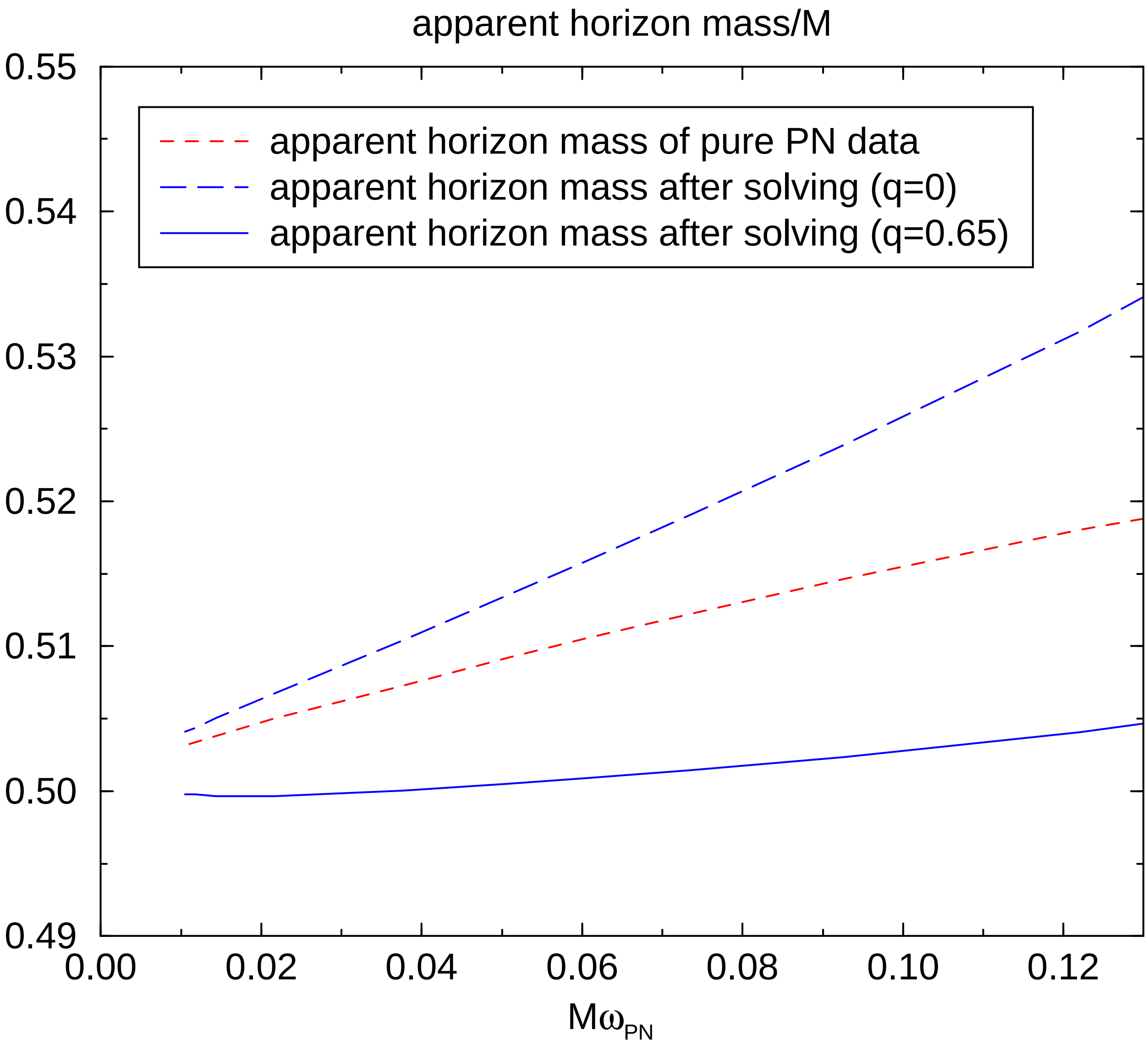
- the change in the ADM mass and the 3-metric induced by solving the constraints with $q = 0.65$ is much smaller than the 2PN corrections
- the change in the extrinsic curvature due to solving is also is much smaller than the 2PN corrections (see plot)





Masses versus the PN angular velocity $\omega_{PN}$

- PN energy has the ISCO minimum near $M\omega_{PN} \approx 0.1$
- ADM mass after solving (with $q = 0.65$) closely follows the PN energy until $M\omega_{PN} \approx 0.05$
- then near $M\omega_{PN} \approx 0.06$ it has the ISCO minimum



- the data after solving with $q = 0.65$ may be close to quasi-equilibrium since the apparent horizon mass is almost constant

# How realistic are the data (with $q = 0.65$)?

For $r_A \gg m_A$ and $r_{12} \gg M$ the data should agree with PN results:

+ this works → the data are realistic in this region

Near the horizon ($r_A \sim m_A$) the data should in principle represent a tidally distorted BH of mass $m_A$:

- Although a Schwarzschild BH of mass $m_A$ without tidal distortion may suffice:

$$\Delta E_{tide} \sim M \left(\frac{M}{r_{12}}\right)^6 \quad \ll \quad \Delta E_{radiated} \approx \frac{m_1 m_2}{2 r_{12}}$$

+ At the puncture the metric approaches the Schwarzschild metric

– BUT near the horizon the data instead are close to puncture data, i.e.

$$g_{ij} \sim \left(1 + \frac{E_1}{2r_1} + \frac{E_2}{2r_2} + u\right)^4 \delta_{ij}$$

which may deviate from Schwarzschild

- How do the data deviate?

+ The apparent horizon mass $m_{AH} \approx m_A$ → the area of the BH seems to be correct

– BUT the shape of the horizon may be wrong?!?!

→ I still need to check how much the horizon is deformed (future work)

## Summary

- We have (for the first time) constructed PN based BH initial data, which do fulfill the constraints
- The extended York procedure (with $q = 0.65$) yields acceptably small changes, so that if the PN data we started with are astrophysically realistic, the data after solving the constraints should still be astrophysically relevant

**Still to do:**

- Examine different mass ratios
- Accurately determine the apparent horizon area and also the angular momentum along the PN inspiral sequence
- Use BH perturbation theory to include the correct tidal distortion of the BHs
- Evolve the data and extract waveforms
- Add spin terms to treat spinning BHs
- Replace the near zone expansion of $h^{TT}_{ij(4)}$ with a globally valid expression

→ If we then evolve numerically we might eventually be able to compute numerical wave forms which continuously match PN wave forms